The Nepali Math. Sc. Report
Vol. 28. No. 1 & 2. 2008 (Dec.)

On ϕ-recurrent Lorentzion para-Sasakian manifolds

DHRUWA NARAIN AND SUNIL KUMAR YADAV

Department of Mathematics and Statistics
D.D.U. Gorakhpur University,
Gorakhpur (273009) India.

Abstract: In this paper we introduce the notion of ϕ-recurrent LP-Sasakian manifolds and show that such a manifold is an Einstein manifold we have also proved that the Charecteristic vector field ξ and the vector field ρ associated to the 1-form A are not co-directional.

Mathematics subject classification (2000) 53C25, 53C05.

Key words and phrases: ϕ-recurrent LP-Sasakain manifold Einstein manifold, sectional curvature.

1. Introduction

In (1989) K. Mstsumoto [4] introduced the notion of LP-Sasakian manifold. I. Mihai and R. Rosca [3] define same notion independently and therefore many authors [4], [5] studied LP-Sasakian manifolds. In (2003) U. C. De, A. A. Shaikh and Sudipta Biswas introduced the notion of ϕ-recurrent Sasakian manifold [1].

In this paper we introduced the notion of ϕ-recurrent LP-Sasakian manifold locally symmeric LP-Sasakian manifold and obtained some interesting results.

2. Preliminaries

A differentiable (smooth) manifold of dimension n is called LP-Sasakian manifold. [2], [3] if it admists a tensor filed ϕ, of type (1.1) a contravariant vector field η, and a covariant vector field η and a Lorentzian metric g which satisfy.

(2.1) $\phi^2 = 1 + \eta \otimes \xi$

(2.2) (a) $\eta(\xi) = -1$ (b) $g(X,\xi) = \eta(X)$ (c) $\eta(\phi X) = 0$ (d) $\phi\xi = 0$

(2.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\,\eta(Y)$

(2.4) (a) $(\nabla_X \phi)(Y) = [\,g(X,Y) + \eta(X)\,\eta(Y)\,]\,\xi + [\,X + \eta(X)\,\xi\,]\,\eta(Y)$

(b) $\nabla_X \xi = \phi X$ (c) $(\nabla_X \eta)(Y) = g(X, \phi Y)$

(2.5) $R(\xi, X)Y = g(X,Y)\,\xi - \eta(Y)X$

(2.6) $R(X,Y)\,\xi = \eta(Y)X - \eta(X)Y$

(2.7) $R(X,\xi)Y = \eta(Y)X - g(X,Y)\,\xi$

(2.8) $\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\,\eta(Y)$

(2.9) $S(X,\xi) = (n-1)\,\eta(X)$

(2.10) $S(\phi, X, \phi Y) = S(X,Y) + (n-1)\,\eta(X)\,\eta(Y)$

for all vector field X, Y, Z where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g, S is the Ricci tensor of type (0, 2) and R is the Riemanian Curvature tensor of the manifold.

Definition (I) *An LP-Sasakian manifold is said to be locally ϕ-symmetnic manifold if $\phi^2((\nabla_W R)(X,Y)Z) = 0$ for all vector field X, Y, Z, W orthogonal to ξ.*

Definition (II) *An LP-Sasakian manifold is said to be a ϕ-recurrent manifold if there exists a non-zero 1-form A such that,*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\;R(X,Y)Z.$$

If the 1–form A vanishes, then the manifold reduces to a ϕ-symmetnic manifold.

3. ϕ-recurrent Lorentzian Para-Sasakian manifolds

Let us consider a ϕ-recurrent LP-Sasakian manifolds then from (2.1) and (2.2) we have,

(3.1) $(\nabla_W R)\,(X,Y)Z + \eta((\nabla_W R)\,(X,Y)Z)\,\xi = A(W)\,R(X,Y)Z$

from which it follows that

(3.2) $g((\nabla_W R)\,(X,Y)Z, U) + \eta((\nabla_W R)\,(X,Y)Z)\,\eta(U) = A(W)\;g(R(X,Y)Z, U)$

let $\{e_i\}$, $i = 1, 2, 3 \ldots n$ be orthnoramal basis of the tangent space at a point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over $i, 1 \le i \le n$, we get

$$(3.3)\qquad (\nabla_W S)(Y,Z) + \sum_{i=1}^{n} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)\, S(Y,Z)$$

The second term of equation (3.3) by putting $Z = \xi$ takes the form

$$(3.4)\qquad g((\nabla_W R)(e_i, Y)Z, \xi)\,\eta(e_i) = g((\nabla_W R)(e_i, Y)\xi, \xi)\, g(e_i, \xi)$$
$$= -g((\nabla_W R)(e_i, Y)\xi, \xi)$$

Since $\quad g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(e_i \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi)$
$$- g(R(e_i, Y)\xi, \nabla_W \xi)$$

and $\{e_i\}$ is an orthonormal basis $\nabla_X e_i = 0$ above equation reduces to

$$(3.5)\qquad g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y) - g(R(\nabla_W \xi.\xi) - g(R(e_i, Y)\,\xi, \nabla_W \xi)$$

using (2.4) and applying the skew symmetric of R, we get

$$(3.6)\qquad g((\nabla_W R)(e_i, Y)\,\xi, \xi) = -g(R(\phi W, \xi)Y, e_i) - g(R(\xi, \phi W)Y, e_i)$$

putting this result in (3.4), we get

$$(3.7)\qquad -g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(\phi W, \xi)Y, e_i) + g(R(\xi, \phi W)Y, e_i)$$

from (3.4), (3.7) and putting $Z = \xi$, we get

$$(3.8)\qquad (n-1)\, g(W, \phi Y) - S(Y, \phi W) = A(W)\eta(Y)$$

replacing Y by ϕY in (3.8) and using (2.1) (2.2) in (3.8), we get

$$S(Y, W) = (n-1)\, g(Y, W)$$

This leads to the following.

Theorem 3.1: *A ϕ-recurrent Lorentzian para-Sasakian manifold M^n is an Einstein manifold.*

Further from (3.1), we have

$$(3.9)\qquad (\nabla_W R)(X, Y)Z = -\eta((\nabla_W R)(X, Y)Z)\,\xi + A(W)\, R(X, Y)Z$$

from (3.9) and the Bianchi identify we have

$$(3.10)\qquad A(W)\,[g(Y,Z)\,\eta(X) - g(X,Z)\eta(Y)] + A(X)\,[g(W,Z)\,\eta(Y)$$
$$-g(Y,Z)\,\eta(W)] + A(Y)\,[g(X,Z)\,\eta(W) - g(W,Z)\,\eta(X)] = 0$$

putting $Y = Z = e_i$ in (3.10) and taking summation over i, $1 \le i \le n$, we have $A(W)\eta(X) = A(X)\eta(W)$ for all vector field X, W replacing X by ξ in (3.11)

$$(3.11)\qquad A(W) = -\rho(\xi)\,\eta(W).$$

where $A(\xi) = g(\xi, \rho) = r(\rho)$, ρ being vector field associated to the I-form A, that is $g(X, \rho) = A(X)$.

Theorem 3.2: *In a ϕ–recurrent Lorentzian para–Sasakian manifold the charecteristics vector field ξ and the vector field ρ associated to the I-form A are not co-directional and I-form A is given by $A(W) = -\eta(\rho)\eta(W)$.*

Since

$$R(X,Y)\phi W = \nabla X \nabla Y \phi W - \nabla Y, \nabla X \phi W - \nabla[X,Y]\phi W. \tag{3.12}$$

Also from $(\nabla_X \phi)(Y) = \nabla_X \phi(Y) - \phi(\nabla_X Y)$ {we get}

$$(\nabla_X \phi)(Y) = (\nabla_X \phi)(Y) + \phi(\nabla_X Y) \tag{3.13}$$

using $(\nabla_X \phi)(Y) = \eta(Y)X + g(X,Y)X + 2\eta(X)\eta(Y)\xi$ and (3.13) in (3.12) we obtained

$$\begin{aligned} R(X,Y)\phi W = {} & \phi R(X,Y)W + g(Y,W)\phi X - g(X,W)\phi Y + g(X, \phi W)Y \\ & - g(Y, \phi W)X + 2\{g(X,\phi W)\eta(Y) - g(Y,\phi W)\eta(X)\xi \\ & + 2\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(W). \end{aligned} \tag{3.14}$$

We now consider that LP-Sasakian manifold (M^n, g) is ϕ-recurrent then from (3.9), we get

$$(\nabla_W R)(X,Y,Z) = -\eta(\nabla_W R)(X,Y,Z)\xi + A(W)R(X,Y,Z) \tag{3.15}$$

and from (3.14) we have

$$\begin{aligned} g((\nabla_W R)(X,Y)\xi, Z)\xi = {} & \{g(X,W)g(\phi, Y, Z) - g(Y,W)g(\phi X, Z)X \\ & + g(\phi R(X,Y)W, Z)\xi \end{aligned}$$

using above this result in (3.15), we get

$$\begin{aligned} g((\nabla_W R)(X,Y,Z) = {} & \{g(X,W)g(\phi Y, Z) - g(Y,W)g(\phi X, Z)\}\xi \\ & + g(\phi R(X,Y)W, Z)\xi + A(W)R(X,Y)Z \end{aligned} \tag{3.16}$$

This leads to the following.

Theorem 3.2: *If an LP-Sasakian manifold is ϕ-recurrent then the relation holds.*

$$\begin{aligned} (\nabla_W R)(X,Y,Z) = {} & \{g(X,W)g(\phi Y, Z) - g(Y,W)g(\phi X, Z)\}\xi \\ & + g(\phi R(X,Y)W, Z)\xi + A(W)R(X,Y)Z \end{aligned}$$

we have more over if the relation (3.16) holds in LP-Sasakian manifold than,

$$\phi^2((\nabla_W R)(X,Y)Z) = \phi^2\{A(W)R(X,Y)Z) \tag{3.17}$$

using (2.1), (2.8) in (3.17) which yeilds

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z)$$

if X and Y are orthogonal to ξ.

We can state the theorem.

Theorem 3.3: *In an LP-Sasakian manifold (M^n, g) satisfying the relation (3.16) is ϕ–recurrent provided that X and Y orthogonal to ξ.*

Let us suppose that in a ϕ-recurrent LP-Sasakian manifold the sectional curvature of a plane $\pi_C T_p(M)$ is defined by

$$k_p(\pi) = g(R(X,Y)Y,X) \tag{3.18}$$

is a non zero constant k, where X, Y is any orthonormal basis of π then from (3.18), we get

$$g((\nabla_z R)(X,Y)Y,X) = 0 \tag{3.19}$$

from (3.9) we get

$$g((\nabla_z R)(X,Y)Y,\xi)\eta(X) = A(Z)\ g(R(X,Y)Y,X). \tag{3.20}$$

Since in a ϕ-recurrent LP-Sasakian manifold the relation (3.16) holds, using (3.16) in (3.20) we get

$$\eta(X)[g(Y,Z)g(\phi X,Y) - g(X,Z)g(\phi Y,X) - g(\phi R(X,Y)Z,Y)] + A(Z)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y)] = kA(Z). \tag{3.21}$$

putting $Z = \xi$ in (3.21) we obtain

$$A(\xi)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y) - k] = 0$$

$$\eta(\rho)\ [g(Y,Y)\eta(X) - g(X,Y)\eta(Y) - k = 0.$$

Which implies that $\eta(\rho) = 0$ then from (3.11) and definition (II) we have

$$\phi^2(\nabla_W R)(X,Y)Z = 0.$$

We state the theorem.

Theorem 3.4: *If a ϕ-recurrent LP-Sasakian has a non zero constant sectional curvature, then it reduces to a locally ϕ-symmetric LP-Sasakian manifold*

REFERENCES

[1] De, U. C., Shaikh, A. A., Sudipta Biswas: On ϕ-recurrent Sasakain manifold NOVISAD. K. Math. Vol. 33. No. 2, 2003, (43–48).

[2] Tripathi, M. M., and De, U. C., On Lorentzian almost para contact and their submanifold, J. Koea, Soc. Math Edu Ser. B. Pure App. Math (2001). 8; (101–105).

[3] Mihai, I., and Rosca, R., On Lorentzian P-Sasakian manifolds classical Analysis, word Scientific public. Singapore (1992) (155–169).

[4] Matsumoto, K., On Lorentzian Para contact manifold, Bull. of Yamagata uni. Nat. Soci, Vol. 12, No. 2, 1989 (151–156).

[5] Matsumoto, K., and Mihai, I.,.: On certain transformation in a Lorentzian Para-Sasakian manifolds; Tensor, N. soci. vol. 47, 1988 (189–197).

The Nepali Math

Vol. 28. No. 1 &

F

set-

Abstract: A related

is obtained. A gene

Keywords and Ph

compact metric sp

2000 Mathematics

1. Introduction

The following theo

Theorem 1.1: *Let*

into Y and let S be

$d($

$d(x', Sy$

$\rho($

$d(x', Sy$