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A Generalized Common Fixed Point in Fuzzy Metric Space

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Abstract: The aim of the present paper is to prove a common fixed point theorem for weakly compatible self mappings in a fuzzy metric space which generalizes and improves various well-known comparable results.

Key Words: Common fixed point, Fuzzy metric space, Weakly compatible maps.

1. Introduction

The study of common fixed points of mappings in a fuzzy metric spacesatisfying certain contractive conditions has been at the center of vigorous research activity. The concept of fuzzy sets was initiated by Zadeh [19] in 1965. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [7]. Grabiec [5] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [17] for a pair of commuting mappings. Also, George and Veeramani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [18] introduced the concept of R-weak commutativity of mappings in fuzzy metric space and Pant [10] introduced the notion of reciprocal continuity of mappings in metric spaces. Also, Jungck and Rhoades [6] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam *et.al* [1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space.

Pant and Jha [11] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam *et. al* [1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space. Recently, Kutukcu *et. al* [8] has established a common fixed point theorem in a fuzzy metric space by studying the relationship between the continuity and reciprocal continuity which is a generalization of the results of Mishra [9] and also gives an answer to the open problem of Rhoades [13] in fuzzy metric space.

The purpose of this paper is to prove a common fixed point theorem for four self mappings in fuzzy metric space under the weak contractive conditions, by relaxing the continuity and reciprocal continuity conditions of mappings and even the completeness. Our result generalizes and improves various other similar results of fixed points. We also give an example to illustrate our main theorem.

We have used the following notions:

Definition 1.1([19]) Let X be any set. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 1.2([4]) A binary operation $*:[0,1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if, $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in $[0, 1]$.

Example: $a * b = ab$, $a * b = \min\{a, b\}$.

Definition 1.3([4]) The triplet $(X, M, *)$ is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all x, y, z in X, $s, t > 0$,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous and $s, t > 0$, In this case, M is called a fuzzy metric on X and the function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. Also, we consider the following condition in the fuzzy metric space $(X, M,*)$:

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.

It is important to note that every metric space (X, d) induces a fuzzy metric space $(X, M, *)$ where $a * b = \min\{a, b\}$ and for all $a, b \in X$, we have $M(x, y, t) = \frac{t}{t + d(x, y)}$, for all $t > 0$, and $M(x, y, 0) = 0$, so-called the fuzzy metric space induced by the metric d.

Definition 1.4([4]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy sequence if, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p > 0$.

A fuzzy metric space $(X, M, *)$ is complete if, every Cauchy sequence in X converges in X.

Definition 1.5([4]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to x in X if, $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

It is noted that since $*$ is continuous, it follows from the condition (iv) of Definition (1.3.) that the limit of a sequence in a fuzzy metric space is unique.

Definition 1.6([1]) Two self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be compatible if, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.7([6]) Two self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if, they commute at coincidence points. That is, $Ax = Sx$ implies that $ASx = SAx$ for all x in X.

It is important to note that a compatible mappings in a metric space are weakly compatible but weakly compatible mappings need not be compatible [16].

Lemma 1.8([14]) Let $(X, M,*)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Lemma 1.9([2]) Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M, *) with the condition (vi) of Definition (1.3). If there exists $k \in (0, 1)$ such that $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $t > 0$ and $n \in \aleph$, the set of natural numbers, then the sequence $\{y_n\}$ is a Cauchy sequence in X.

If A, B, S and T are self mappings of fuzzy metric space $(X, M, *)$ in the sequel, we shall denote

$$N(x, y, t) = M(Ax, Sx, t) * M(By, Ty, t) * M(Sx, Ty, t) * M(Ax, Ty, \alpha t) * M(Sx, By, (2-\alpha), t),$$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

2. Main Result

Theorem 2.1. Let $(X, M, *)$ be a fuzzy metric space with additional condition (vi) and with $a * a \geq a$ for all $a \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that

(i) $AX \subseteq TX$, $BX \subseteq SX$,

(ii) there exists $k \in (0, 1)$ such that $M(Ax, By, kt) \geq N(x, y, t)$,

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

If one of AX, BX, SX and TX is complete subspace of X and if the pair (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Then, since $AX \subseteq TX$, $BX \subseteq SX$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \text{ for } n = 0, 1, 2, \ldots.$$

Now, putting $x = x_{2n}$, $y = x_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (ii), we have

$$M(Ax_{2n}, Bx_{2n+1}, kt) \geq M(Ax_{2n}, Sx_{2n}, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n+1}, (1-q)t) * M(Sx_{2n}, Bx_{2n+1}, (1+q)t).$$

That is,

$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, (1+q)t) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, qt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, qt)$.

Since t – norm * is continuous, letting $q \to 1$, we have

$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, t) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$.

It follows that, $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}\}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$.

Similarly, $M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$. Therefore, for all n even or odd, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}\}, y_n, t) * M(y_n, y_{n+1}, t).$$

Consequently, $M(y_n, y_{n+1}, t) \geq M(y_{n-1}\}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-1}t)$.

Using a simple induction, we have

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-m}t).$$

Since $M(y_n, y_{n+1}, k^{-m}t) \to 1$ as $m \to \infty$, it follows that

$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$, for all $n \in$ N and $t > 0$. Therefore, by Lemma (1.8.), the sequence $\{y_n\}$ is a Cauchy sequence.

Now, suppose TX is a complete subspace of metric space X. Then, the subsequence $y_{2n} = Tx_{2n+1}$ is also a Cauchy sequence in TX and hence has a limit u. Let $v \in T^{-1}u$, then $Tv = u$. Since y_{2n} is convergent, so y_n is convergent to u and hence y_{2n+1} also converges to u.

Now, setting $x = x_{2n}$ and $y = v$ in (ii), with $\alpha = 1$, we have

$M(Ax_{2n}, Bv, kt) \geq M(Ax2n, Sx_{2n}, t) * M(Bv, Tv, t) * M(Sx_{2n}, Tv, t) * M(Ax_{2n}, Tv, t) * M(Sx_{2n}, Bv, t)$,

letting $n \to \infty$, we have $M(u, Bv, kt) \geq M(u, Bv, t)$. This implies that $Bv = u$. Also, since $BX \subseteq SX$, so

$u = Bv$ implies that $u \in SX$. Let $w \in S^{-1}$u, then $Sw = u$. Again, setting $x = w$ and $y = x_{2n+1}$ in (ii) with

$\alpha = 1$, we get

$M(Aw, Bx_{2n+1}, kt) \geq M(Aw, Sw, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Sw, Tx_{2n+1}, t) * M(Aw, Tx_{2n+1}, t) * M(Sw, Bx_{2n+1}, t)$.

Letting $n \to \infty$, we get $M(Aw, u, kt) \geq M(Aw, u, t)$, which implies that $u = Aw$.

Thus, we have $u = Tv = Bv = Aw = Sw$ ... (1)

Now, since $u = Tv = Bv$, so by the weak compatibility of (B, T), it follows that $BTv = TBv$ and so we get $Bu = BTv = TBv = Tu$. Also, since $u = Sw = Aw$, so by the weak compatibility of (A, S), it follows that $ASw = SAw$ and so we have $Au = ASw = SAw = Su$. Thus, from (ii) with $\alpha = 1$, we have

$M(Aw, Bu, kt) \geq M(Aw, Sw, t) * M(Bu, Tu, t) * M(Sw, Tu, t) * M(Aw, Tu, t) * M(Sw, Bu, t)$,

That is, $M(u, Bu, kt) \geq M(u, Bu, t)$, which is a contradiction. This implies that $u = Bu$. Similarly, using (ii) with $\alpha = 1$, one can show that $Au = u$. Therefore, we have $u = Bu = Tu = Au = Su$. Hence, the point u is a common fixed point of A, B, S and T. If we assume SX is complete, then the argument analogue to the previous completeness argument proves the theorem. If AX is complete, the $u \in AX \subseteq TX$. Similarly, if BX is complete, then $u \in BX \subseteq SX$.

The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (ii). In fact, if u' be another fixed point for mappings A, B, S and T. Then, for $\alpha = 1$, we have

$M(u, u', kt) = M(Au, Bu', kt) \geq M(Au, Su, t) * M(Bu', Tu', t) * M(Su, Tu', t) * M(Au, Tu', t) * M(Su, Bu', t), \geq M(u, u', t)$, and hence, we get $u = u'$.

This completely establishes the theorem.

We now give an example to illustrate the above theorem.

Example: Let $X = [2, 20]$ and M be the usual fuzzy metric space on $(X, M, *)$. Define A, B, S and $T : X \to X$ as follows:

$A2 = 2$, $Ax = 3$ if, $x > 2$;

$Bx = 2$ if, $x = 2$ or > 5, $Bx = 6$ if, $2 < x \leq 5$;

$S2 = 2$, $Sx = 6$ if, $x > 2$;

$T2 = 2$, $Tx = 12$ if, $2 < x \leq 5$, $Tx = x - 5$ if, $x > 5$.

Also, we define $M(Ax, By, t) = \frac{t}{[t + d(x, y)]}$, for all x, y in X and for all $t > 0$. Then, for $\alpha = 1$, the

pair (A, S) and (B, T) are weakly compatible mappings. Also, these mappings satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$.

Remarks: As the earlier fixed point theorems have been established using stronger contractive conditions, so our results generalizes the results of Kutukcu *et.al*[8] and that of Sharma [14], Mishra [9]. Consequently, it improves and unifies the results of Balasubramaniam *et.al* [1], Chugh and Kumar

[3], Pant and Jha [11], Pant [12], Sharma *et.al* [15] and other similar results for fixed points.

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