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A Generalization Of Unified Common Fixed Point Theorem

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Abstract: The aim of the present paper is to obtain a common fixed point theorem for a set of four mappings, which includes all the known contractive definitions as particular cases and employs a Lipschitz type analogue of known contractive definitions. Also we provide a new type of answer to the open problem posed by Rhoades [21] on the existence of a contractive definition.

Key Words and Phrases : Common fixed point, compatible mappings, contractive conditions.

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has attracted a great deal of research activity during the last two decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form

(1) $$d(Ax, By) \leq h\, m(x, y),\ 0 \leq h < 1,$$

where $m(x,y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$, or, a Meir-Keeler type (ε, δ)-contractive condition of the form given $\varepsilon > 0$ there exists a $\delta > 0$ such that

(2) $$\varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon,$$

or, a ϕ-contractive condition of the form

(3) $$d(Ax, By) \leq \phi(m(x,y)),$$

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ-contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) ϕ (t) is non decreasing and $t/(t - \phi(t))$ is non increasing ([2]),

(II) ϕ (t) in non decreasing and $\lim_n \phi^n (t) = 0$ for each $t > 0$([4], [9]),

(III) ϕ is upper semi continuous ([1], [4], [8], [14]) or equivalently,

(IV) ϕ is non decreasing and continuous from right ([20]).

It is now known (e.g. [4], [16]) that if any of the conditions (I), (II), (III), or (IV) is assumed on ϕ, then a ϕ-contractive condition (3) implies an analogous (ε,δ)-contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir-Keeler type (ε,δ)-contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ε,δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3).

Example 1. ([16]) Let $X = [0, 2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by $fx = (1 + x) / 2$ if $x < 1$; $fx = 0$ if $x \geq 1$. Then, it satisfies contractive condition $\varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\} < \varepsilon + \delta$ $\Rightarrow d(fx , fy) < \varepsilon$; with $\delta (\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta (\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point. Also f does not satisfy the contractive condition

$$d(fx, fy) \leq \phi (\max \{d(x,y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)] /2\}),$$

since the desired function $\phi(t)$ cannot be defined at $t = 1$.

Hence, the two type of contractive conditions (2) and (3) are independent of each others. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following condition on the function δ have been introduced and used by various authors.

(V) δ is non decreasing ([12], [14]),

(VI) δ is lower semi continuous ([6], [7]).

Jachymski [4] has shown that the (ε ,δ)-contractive condition (2) with a non decreasing δ implies a ϕ-contractive condition (3). Also, Pant *et al.* [16] have shown that the (ε , δ)-contractive condition (2) with a lower semi continuous δ, implies a ϕ-contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the (ε , δ)-contractive condition (2) implies an analogous ϕ-contractive condition (3) and both the contractive conditions hold simultaneously.

It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with additional condition on δ or ϕ respectively. Besides the contractive condition (2), the ϕ-contractive condition is also assumed simultaneously with or even without imposing any additional restriction either on ϕ or on δ [13,16]. Such theorems not only unify the Meir-Keeler type fixed point theorem and Boyd-Wong type fixed point but also improve them. A more general approach of generalizing these results consists of assuming Lipschitz type analogue of contractive condition. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and, we assume contractive condition (2) together with a Lipschitz type analogue of condition (3); that is, a condition of the form

$$d(Ax, By) < \max \{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2, [d(Sx, By) + d(Ax, Ty)]/2\}, \; 1 \leq k < 2.$$

We prove a common fixed point theorem for four mappings using this approach. It may be noted that all the known results have been dealt with the case $k = 1$ and so all such results are obtained as special case of our theorem.

Two self-mappings A and S of a metrix space (X, d) are called *compatible* (see Jungck [6]) if, $\lim_n d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X. It is easy to see that compatible maps commute at their coincidence points.

2. Results

We prove the following theorem with the notation M(x, y) defined as

$$M(x, y) = \max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}.$$

Theorem 1. *Let* (A, S) *and* (B, T) *be compatible pairs of self mappings of a complete metric space* (X, d) *such that*

i) $AX \subset TX$, $BX \subset SX$,

ii) given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \leq M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$, and

iii) $d(Ax, By) < \max \{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2 \; [d(Sx, By) + d(Ax, Ty)]/2\}, \; 1 \leq k < 2.$

If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.

Proof : Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

(1.1) $$y_{2n} = Ax_{2n} = Tx_{2n+1} \; ; \; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

This can be done by virtue of (i). We claim that $\{y_n\}$ is a Cauchy sequence.

Using (ii), we get $d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \leq M(x_{2n}, x_{2n+1}) < d(y_{2n-1}, y_{2n})$.

Similarly, we get $d(y_{2n-1}, y_{2n}) < d(y_{2n-2}, y_{2n-1})$ and so on.

Thus, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \geq 0$. If possible, suppose $r > 0$. Then given $\delta > 0$ there exists a positive integer N such that for each $n \geq N$, we have

(1.2) $$r < d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \leq M(x_{2n}, x_{2n+1}) < r + \delta.$$

Selecting δ in (1.2) in accordance with (ii), for each $n \geq N$, we get

$d(y_{2n+2}, y_{2n+1}) = d(Ax_{2n+2}, Bx_{2n+1}) < r$. This, in turn, gives

$d(y_{2n+3}, y_{2n+2}) < d(y_{2n+1}, y_{2n+2}) < r$. This contradicts (1.2). Hence, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We now show that $\{y_n\}$ is a Cauchy sequence.

Suppose it is not, then there corresponds an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) > 2\varepsilon$. Selecting δ in (ii), so that $0 < \delta < \varepsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, so there exists an integer N such that $d(y_n, y_{n+1}) < \delta / 6$ whenever $n \geq N$. Let $n_i \geq N$ then there exists integer m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) > \varepsilon + (\delta / 3)$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) < \varepsilon + (\delta / 3) + (\delta / 6) < 2\varepsilon,$$

which is a contradiction. Now, without loss of generality, we can assume n_i to be odd.

Let m_i be the smallest even integer such that $d(y_{n_i}, y_{m_i}) > \varepsilon + (\delta / 3)$. Then $d(y_{n_i}, y_{m_i-2}) < \varepsilon + (\delta / 3)$ and

$$\varepsilon + (\delta / 3) < d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i})$$
$$< \varepsilon + (\delta / 3) + (\delta / 6) + (\delta / 6) = \varepsilon + (2\delta / 3);$$

that is,

(1.3) $$\varepsilon + (\delta / 3) < d(y_{n_i}, y_{m_i}) \leq \varepsilon + (2\delta / 3).$$

Also, using (ii), we get $d(y_{n_i+1}, y_{m_i+1}) \leq M(x_{n_i+1}, x_{m_i+1}) < \varepsilon + (2\delta / 3) + (\delta / 6) < \varepsilon + \delta$;

that is, $\varepsilon + (\delta / 3) < M(x_{n_i+1}, x_{m_i+1}) < \varepsilon + \delta$. Using (ii), we get $d(y_{n_i+1}, y_{m_i+1}) < \varepsilon$.

But then, $$d(y_{n_i}, y_{m_i}) \leq d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i})$$
$$< (\delta / 6) + \varepsilon + (\delta / 6) = \varepsilon + (\delta / 3),$$

which contradicts (1.3). Hence $\{y_n\}$ is a Cauchy sequence in X. But X is complete so there exists a point z in X such that $y_n \to z$. Also, using (1.1), we have

(1.4) $$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Suppose that S is continuous. Then $SSx_{2n} \to Sz$, $SAx_{2n} \to Sz$ and compatibility of A and S implies that $ASx_{2n} \to Sz$. Also, since $AX \subset TX$, corresponding to each value of n, there exists z_{2n} in X such that $ASx_{2n} = Tz_{2n}$. Thus,

$ASx_{2n} = Tz_{2n} \to Sz$ and $SSx_{2n} \to Sz$. We show that $\lim_n Bz_{2n} = Sz$. If not, then there exists a subsequence $\{Bz_{2m}\}$ of $\{Bz_{2n}\}$, a number $r > 0$ and a positive integer N such that for each $m > N$, we have $d(ASx_{2m}, Bz_{2m}) \geq r$, $d(Sz, Bz_{2m}) \geq r$ and in view of (iii), we get

$$d(ASx_{2m}, Bz_{2m}) < \max\{d(SSx_{2m}, Tz_{2m}),\ k[d(ASx_{2m}, SSx_{2m}) + d(Bz_{2m}, Tz_{2m})]/2,\ [d(SSx_{2m}, Bz_{2m}) + d(ASx_{2m}, Tz_{2m})]/2\},$$

which, on letting $m \to \infty$, yields

$$d(Sz, Bz_{2m}) \leq k[d(Sz, Bz_{2m})]/2 < d(Sz, Bz_{2m}), \text{ a contradiction.}$$

Hence $\lim_{n\to\infty} Bz_{2n} = Sz$. We claim that $Az = Sz$. If $Az \neq Sz$, then by virtue of (iii), for sufficiently large values of n, we get

$$d(Az, Bz_{2n}) < \max\{d(Sz, Tz_{2n}),\ k[d(Az, Sz) + d(Bz_{2n}, Tz_{2n})]/2,\ [d(Sz, Bz_{2n}) + d(Az, Tz_{2n})]/2\},$$
$$= k[d(Az, Bz_{2n})]/2.$$

On letting $n \to \infty$, this yields $d(Az, Sz) < k[d(Az,Sz)]/2 < d(Az, Sz)$, a contradiction. Hence $Az = Sz$. Again, since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. If $Bw \neq Tw$, using (iii) we get

$$d(Az, Bw) < \max\{d(Sz, Tw),\ k(d(Az, Sz) + d(Bw, Tw)]/2,\ [d(Sz, Bw) + d(Az, Tw)]/2\}.$$
$$= k[d(Bw, Tw)]/2 < d(Bw, Tw) = d(Bw, Az), \text{ a contradiction.}$$

Hence $Az = Bw$ and so, $Sz = Az = Tw = Bw$.

Since compatible maps commute at their coincidence points, we get $ASz = SAz$ and $BTw = TBw$. Moreover, $AAz = ASz = SAz = SSz$ and $BBw = BTw = TBw = TTw$.

If $Az \neq AAZ$, using (iii), we find

$$d(Az, AAz) = d(AAz, Bw) < \max\{d(SAz, Tw),\ k[d(AAz, SAz) + d(Bw, Tw)]/2,\ [d(SAz, Bw) + d(AAz, Tw)]/2\}.$$
$$= k[d(Bw, AAz)]/2 < d(AAz, Bw), \text{ a contradiction.}$$

So that $Az = AAz = SAz$ and so Az is a common fixed point of A and S. Similarly, Bw (=Az) is a common fixed point of B and T. Uniqueness of the common fixed point follows from (ii). The proof is similar when T is assumed continuous in place of S. Moreover, since $AX \subset TX$ and $BX \subset SX$, the proof follows on similar lines when A or B is assumed to be continuous. This establishes the theorem.

We now give an example to illustrate the above theorem.

Example 2. Let $X = [2, 20]$ and d be the Euclidean metric on X. Define A, B, S and $T: X \to X$ as follows :

$Ax = 2$ for each x:

$Bx = 2$ if $x < 4$ or ≥ 5,	$Bx = 3 + x$	if $4 \leq x < 5$;
$Sx = x$ if $x \leq 8$.	$Sx = 8$	if $x > 8$;
$Tx = 2$, if $x < 4$ or ≥ 5,	$Tx = 9 + x$	if $4 \leq x < 5$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be seen in this example that A, B, S and T satisfy the condition (ii) when $\delta(\varepsilon) = 1$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6 - \varepsilon$ if $\varepsilon < 6$. Thus, $\delta(\varepsilon)$ is neither non decreasing nor lower semi continuous. It can also be verified that the mappings A, B, S and T do satisfy the contractive condition (iii) with $k = 1$. However, A, B, S, and T do not satisfy the ϕ-contractive condition (3) since the required function $\phi(t)$ can not be defined at $t = 6$. Hence we see that the present example does not satisfy the conditions of any previously known common fixed point theorem for contractive type mappings, since neither the mappings satisfy a ϕ-contractive condition nor δ is lower semi continuous or non decreasing.

Now, as a corollary of Theorem 1, we obtain the following Theorem 2, which provides a new type or affirmative answer to an open problem (e.g. see Rhoades [21], p. 242) on the existence of a contractive definition, which is strong enough to generate a fixed point but does not force the map to be continuous at their common fixed point. It may be observed in this context that fixed point theorems either explicitly assume continuity of mappings or, as shown by Rhoades [21] and Hicks and Rhodes [3], the contractive definitions themselves imply continuity at the fixed point. This makes, the next theorem an interesting result.

Theorem 2. *Let f be a self mapping of a complete metric space* (X, d) *such that for any* x, y *in* X (i) *given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq \max \{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(fx, y)]/2\} < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon,$$

and

(ii) $d(fx, fy) \leq \max \{d(x, y), k[d(x, fx) + d(y, fy)] / 2, [d(x, fy) + d(fx, y)] / 2\}$,

where $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$. Then f has a unique fixed point.

Proof : Theorem 2 follows from Theorem 1 by taking $S = T = I_X$, an identity mapping on X and $A = B = f$, together with $k = 1$ in Theorem 1.

It may also be noted that f need not be continuous in Theorem 2, as illustrated in the next example.

Example 3. Let $X = [0, 10]$ and d is the usual metric on X. Define $f : X \to X$ as follows: $fx = (x + 5)/2$ if $x \leq 5$ and $fx = 0$ if $x > 5$. Then f satisfies all the conditions of the above Theorem 2 and has a unique fixed point $x = 5$. It can be verified that conditions (i) and (ii) of Theorem 2 are satisfied with $\delta(\varepsilon) = 5$ if $\varepsilon \geq 5$ and $\delta(\varepsilon) = 5 - \varepsilon$ if $\varepsilon < 5$ when $k = 1$. It can be seen that f is discontinuous at the fixed point $x = 5$.

Remarks : As various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so our Theorem 1 improves the results of Boyd and Wong [1], Carbone et al. [2], Matkowski [9], Pant [11, 12, 15], Pant and Pant [13, 14], Park and Rhoades [20], Singh and Kashhara [23], Jungck [6], Jungck *et al.* [7]. Jachymski [4, 5], Maiti and Pal [8], and Park and Bae [19] for the case when $k = 1$. All such results are obtained as special case of Theorem 1 when $k = 1$. Also our theorem, thus, generalizes the results of Pant [15], Pant and Pant [13, 14], Pant *et al.* [16], Pant and Jha [17, 18] and all other similar results for fixed points and allows k to take values other than 1 by taking a Lipschitz type contractive condition. Moreover Theorem 2 provides a new type of affirmative answer to the open problem since our theorem assumes Lipschitz type analogue of a plane contractive condition instead of a ϕ-contractive condition.

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