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A Note Concerning the Invariance of Baire Spaces under Mappings

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Abstract: In this note we prove that under a semi-continuous and almost open mapping the image of a Baire space is also a Baire space and as a result improves a theorem of Dasgupta and Lahiri [3]. Furthermore, a theorem of Noiri [6] on irresolute mapping is improved in this process.

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1. INTRODUCTION

It is well known that under a continuous and open mapping, the image of a Baire space is also a Baire space. Dasgupta and Lahiri [3] and Frolik [4] reached the same conclusion under weaker hypotheses. In this note we prove another generalization of this classical theorem which improves the result of Dasgupta and Lahiri [3] and is independent from that of Frolik [4]. In this process we also improve a theorem of Noiri (Theorem 3 of [6]) which tells when a mapping is irresolute. Throughout the paper X, Y denote topological spaces, ϕ the empty set, $\Re$ the set of real numbers and U the usual topology. The closure and the interior of a set $A \subset X$ is denoted by Int A and Cl A respectively.

2. PRELIMINARIES

Definition 1.1 Let $A \subset X$. Then

(a) A is said to be semi-open [5] iff there exists an open set O such that $O \subset A \subset ClO$, or equivalently, $A \subset Cl\text{Int}\, A$. The union of all semi-open sets contained in A is called the semi-interior [3] of A and is denoted by SInt A (b) A is said to be semi-closed [3] iff the complement of A, $X - A$ is semi-open. The intersection of all semi-closed sets containing A is called the semi-closure [3] of A and is denoted by SCl A.

Remark 1.1 It is known from [3] that SCl A is semi-closed and SInt A is semi-open.

Definition 1.2 Let $f: X \to Y$ be a mapping. Then

(a) f is called semi-continuous [5] if for each open set V in Y, $f^{-1}(V)$ is semi-open in X.

(b) f is called almost open (in the sense of Rose) [7] if $f(U) \subset \text{Int}\, Cl\, f(U)$ for every open set U in X.

(c) f is called almost continuous [4] if for every open subset V of Y, $f^{-1}(V) \subset Cl\, \text{Int} f^{-1}(V)$.

(d) f is called feebly open [4] if $A \subset X$, $\text{Int}\, A \neq \phi \Rightarrow f(\text{Int}\, A) \neq \phi$.

Remark 1.2

(a) The notions of semi-continuity [5] and almost continuity [4] are the same in view of Definition 1.1 (a).

(b) Every continuous mapping is semi-continuous but the converse is not necessarily true (see [5]).

(c) Every open mapping is almost open (feebly open) (see [7], ([4])), but the converse is not necessarily true (see Example 2.2 and (Example 2.1)).

Theorem 1.1 [3] A set $A \subset X$ is semi-open iff SInt A = A and A is semi-closed iff $SCl\, A = A$.

Theorem 1.2 [3] If $A \subset X$, then $\text{Int}\, A \subset \text{SInt}\, A \subset A \subset SCl\, A \subset ClA$.

Theorem 1.3 [5] If A is a semi-open subset of X and $B \subset X$ such that $A \subset B\; Cl\, A$, then B is semi-open in X.

Lemma 1.1 [3] A set $A \subset X$ is semi-closed iff there exists a closed set F such that $\text{Int}\, F \subset A \subset F$.

Lemma 1.2 [3] For any $A \subset X$, $\text{SInt}\, A = X - [SCl\,(X - A)]$

Theorem 1.4 [7] A mapping $f: X \to Y$ is almost open iff $f^{-1}(ClV) \subset Cl\, f^{-1}(V)$, for every open $V \subset Y$.

Theorem 1.5 [3] Let $f: X \to Y$ be surjective, open and semi-continuous. If X is a Baire space then Y is also a Baire space.

Theorem 1.6 [4] Let $f: X \to Y$ be surjective, feebly open and almost continuous. If X is a Baire space then Y is also a Baire space.

3. IMPROVEMENT OF CERTAIN THEOREMS

We start this section from the following theorem of Noiri [6].

Theorem 2.1 (Theorem 3 of [6]. If $f: X \to Y$ is an open and semi-continuous mapping then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X. Our purpose is to set this result in a more general context. More precisely, we shall prove that the openness of the mapping can be replaced by the almost openness of the mapping and henceforth making an improvement (in view of Remark 1.2 (c)) of the above theorem.

Theorem 2.2 If $f: X \to Y$ is an almost open and semi-continuous mapping, then the inverse image $f^{-1}(B)$ of each semi-open set B in Y is semi-open in X.

Proof. Let B be an arbitrary semi-open set in Y. Then there exists an open set V in Y such that $V \subset B \subset Cl\, V$. Since f is almost open, we have, $f^{-1}(ClV) \subset Cl\, f^{-1}(V)$, by Theorem 1.4. Again since f is semi-continuous, $f^{-1}(V)$ is semi-open in X. Thus $f^{-1}(V) \subset f^{-1}(B) \subset Cl\, f^{-1}(V)$ and so $f^{-1}(B)$ is semi-open in X, by Theorem 1.3.

Before going to our main theorem (Theorem 2.6) we shall prove some ancillary results in the following.

Theorem 2.3 Let $f: X \to Y$ be surjective and almost open. Then if $B \subset Y$ is dense and open in Y, $f^{-1}(B)$ is dense in X.

Proof. Let B be dense and an open subset of Y. Then since f is almost open, by Theorem 1.4 we have,

$$f^{-1}(Cl\, B) \subset Cl\, f^{-1}(B) \Rightarrow f^{-1}(Y) \subset Cl\, f^{-1}(B) \Rightarrow X \subset Cl\, f^{-1}(B) \Rightarrow f^{-1}(B) \text{ is}$$

dense in X.

Theorem 2.4 $A \subset D$ is dense in X iff $SCl\, D = X$.

Proof. Let D be dense in X so that by Theorem 1.2, $SClD \subset ClD = X$. Since SCl D semi-closed by Remark 1.1, there is a closed set $F \subset X$ such that $Int\, F \subset SCl\, D$

$\subset$ F by Lemma 1.1. But by Theorem 1.2, D $\subset$ SCl D $\subset$ F and so X = Cl D $\subset$ Cl F. Thus F = X and we get X = Int X $\subset$ SCl D $\subset$ X, which implies that SCl D = X.

If SCl D = X, then because by Theorem 1.2, SCl D $\subset$ Cl D, it follows that Cl D = X and so D is dense in X.

Remark 2.1 The above theorem is an improvement of the requirement that D is dense iff Cl D = X.

Theorem 2.5 A set D is dense in X iff the complement of D has empty semi-interior.

Proof. If D is dense in X, then by Theorem 2.4, SCl D = X and so in Lemma 1.2, replacing A by X – D we obtain, SInt (X – D) = X – SCl D = ϕ. On the other hand if SInt (X – D) = ϕ, then we obtain X – SCl D = SInt (X – D) = ϕ and so SCl D = X. Hence D is dense in X, by Theorem 2.4

Now we are in a position to show that a Baire space remains invariant under an almost open, semi-continuous surjection.

Theorem 2.6 Let f : X $\rightarrow$ Y be surjective, almost open and semi-continuous. If X is a Baire space then Y is also a Baire space.

Proof. Let G = $\cap_i$ D_i be a countable intersection of dense open sets in Y.

As f is almost open and semi-continuous by Theorem 2.2, f^{-1} (SInt (X – G)) is semi-open in X. Thus

$$f^{-1}(\text{SInt}(X - G)) = \text{SInt} f^{-1} (\text{SInt}(X - G)), \text{ by Theorem 1.1}$$
$$\subset \text{SInt} f^{-1} (X - G), \text{ by Theorem 1.2}$$
$$= \text{SInt} f^{-1} (X - \cap_i D_i)$$
$$= \text{SInt} f^{-1} [X - \cap_i f^{-1} (D_i)] \qquad (1)$$

Since D_i is open for each i, semi-continuity of f implies that f^{-1} (D_i) is semi-open in X for each i and so there is an open set V_i such that $V_i \subset f^{-1} (D_i) \subset Cl\, V_i$, for each i. Thus X - $\cap_i f^{-1}$ (D_i) $\subset$ X - $\cap_i$ V_i. So from (1) we get,

$$f^{-1} [\text{SInt}(X - G)] \subset \text{SInt}[X - \cap_i V_i] \qquad (2)$$

Now by Theorem 2.3, f^{-1} (D_i) is dense in X. So ClV_i = X. Consequently V_i is dense in X for each i. Thus $\cap_i V_i$ is a countable intersection of dense open set in X, and as X is a Baire space, $\cap_i V_i$ is dense in X so that SInt [X - $\cap_i V_i$] = ϕ, by Theorem 2.5. Hence from (2) we get, f^{-1} [SInt (X – G)] = ϕ, so that SInt (X – G) = ϕ, which implies, by Theorem 2.5, that G is dense in X. Thus Y is a Baire space.

Remark 2.2 From Remark 1.2 (c) it follows that Theorem 2.6 is an improvement of Theorem 1.5. Furthermore, Theorem 2.6 of ours and Theorem 1.6 of Frolik does not imply any one from the other, because although by Remark 1.2 (a) the semi-continuity of Levine and almost continuity of Frolik are the same but feebly open mapping and almost open mapping of Rose are independent to each other as shown in the following examples.

Example 2.1

Let $f : (\Re, U) \rightarrow (\Re, U)$ be defined by $f(2) = 4$ and $f(x) = x$ otherwise. It is easy to verify that f is feebly open. But considering the open set $U = (1, 3)$ we see $f(U) = (1, 2) \cup (2/ 3) \cup (2, 3) \cup \{4\}$. Hence $Cl\, f(U) = [1. 3] \cup \{4\}$ and so $\text{Int}\, Cl\, f(U) = (1, 3) \supseteq f(U)$. Therefore f is not almost open.

Example 2.2

Let N be the set of natural numbers and τ be the topology consisting of all sets O such that $O = \phi$ or $O = N$ or $O = \{1, 2, ..., n\}$ for each $n (> 1)$ in N. Let $f: (\Re, U) \rightarrow (N, \tau)$ be defined by

$$f(x) = 1, \text{ if x is rational}$$
$$= 3, \text{ if x is irrational and } 0 < |x| < 3$$
$$= n, \text{ if x is irrational and } n - 2 < |x| < n - 1 \text{ where } n \in N \text{ and } > 4.$$

Let O be any open set in $(\Re, U)$. Then $f(O)$ must contain 1. So $\text{Int}Cl\, f(O) = \text{Int}\, N = N \supset f(O)$. Hence f is almost open. Next we consider the open set $O = (0. 1)$ in $(\Re, U)$. Clearly $\text{Int}\, O \neq \phi$. But $f(O) = \{1, 3\}$ and so $\text{Int}\, f(O) = \phi$. Hence f is not feebly open.

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