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A note on common fixed point principle

K. JHA
Faculty of Mathematics, Department of Natural Sciences,
School of Science, Kathmandu University,
P. O. Box No. 6250, Kathmandu, Nepal.
E-mail: jhakn@ku.edu.np

R. P. PANT
Department of Mathematics, Kumaon University,
D. S. B. Campus, Nainital–263 002, Uttaranchal, India.

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last three decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form,

(1) $$d(Ax, By) \le h\, m(x,y), \qquad \text{for } 0 \le h < 1,$$

where $m(x,y) = \max\,\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ [d(Sx,By) + d(Ax,Ty)]/2\}$,
or, a Meir–Keeler type (ε,δ)-contractive condition of the form,

given $\varepsilon > 0$, there exists a $\delta > 0$ such that

(2) $$\varepsilon \le m\,(x,y) < \varepsilon + \delta \Rightarrow \mathrm{d}(Ax, By) < \varepsilon,$$

or, a ϕ-contractive condition of the form,

(3) $$d\,(Ax, By) \le \phi\,(m(xy)),$$

involving a contractive gauge function $\phi : \Re_+ \to \Re_+$ is such that $\phi(t) < t$ for each $t > 0$. Also, the weak form of contractive condition (2) is of the form,

given $\varepsilon > 0$, there exists a $\delta > 0$ such that

(4) $$\varepsilon < m(x,y) < \varepsilon + \delta \Rightarrow d(Ax, By) \le \varepsilon.$$

Clearly, condition (1) is a special case of both conditions (2) and (3). Moreover, Jachymski [2] has shown that contractive condition (2) implies (4) but the contractive condition (4) does not imply the contractive condition (2). Pant [4] proved the following common fixed point theorem.

Theorem 2.1 [4]: *Let (A, S) and (B, T) be compatible pairs of self maps of a complete metric space (X,d) and such that $AX \subset TX$, $BX \subset SX$ and*

(i) *given* $\varepsilon > 0$, there exists a $\delta > 0$ *such that*

$$\varepsilon < \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ [d(Sx,By) + d(Ax,Ty)]/2\} \leq \varepsilon + \delta$$
$$\Rightarrow d(Ax,By) \leq \varepsilon,$$

(i)* $d(Ax,By) < \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ [d(Sx,By)+d(Ax,Ty)]/2\}$ *whenever the right hand side is positive,*

(ii) $d(Ax,By) \leq \max\{d(Sx,Ty),\ d(Ax,Sx),\ d(By,Ty),\ k[d(Sx,By)+d(Ax,Ty)]/2\}$ *where* $0 \leq k < 2$ *and* $\phi: \Re_+ \to \Re_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$.

If one of the mappings A, B, S or T is continuous, then A, B, S and T have a unique common fixed point.

This theorem gives a more general approach of generalizing the known common fixed point theorems for four mappings by assuming slightly weaker form of a Meir-Keeler type contractive condition together with a Lipschitz type contractive condition of the form (ii).

The main objective of this note is to provide a correction to a minor error in the proof of Theorem 2.1 of [4]. On page 293 (line 16–20) of Pant [4], condition (ii) has been used to arrive at the contradiction $d(AAz, Bw) < d(AAz, Bw)$.

However, by using condition (ii), we can obtain the above contradiction only if, $k \leq 1$ and not for $k > 1$. The correct approach to arrive at the desired contradiction would be to use condition (i)* in place of condition (ii) and then replace lines 16 – 20 on page 293 by the following derivation.

If $Az \neq AAz$, then using (i)*, we get

$$\begin{aligned} d(Az, AAz) &= d(AAz, Bw) \\ &< \max\{d(SAz, Tw),\ d(AAz, SAz),\ d(Bw, Tw), \\ &\qquad [d(AAz, Tw) + d(Bw, SAz)]/2\} \\ &= d(AAz, Bw), \end{aligned}$$

which is a contradiction.

The remaining part of proof of Theorem 2.1 of [4] remains unaltered.

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