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A note on the upper radical of $S(\rho_1+\rho_2)$ of hemirings

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Abstract: In this paper, we generalize a few results of [4] for the upper radical classes of rings for radical classes of hemirings, by using the construction for the upper radical classes of hemirings.

1. Introduction and Preliminaries

The notion of radical classes of hemirings was introduced by D. M. Olson and T. L. Jenkins [7], as an extension of radical classes of rings (see [4]). The theory was further enriched by many authors (see [7, 8]).

Y. Lee and R. E. Propes [4] introduced the concept of the sum of two radical classes of rings. They have shown that 'sum' is not a radical class in general. In the present paper, we extend the notion of sum of two radical classes of hemirings and generalize a few results of [4] in the framework of hemirings. The sum of two radical classes was investigate by [4] for radical classes of rings. Here we are interesting to generalize a few results of [4] in the framework of hemiring which is quite different from ring theoretical approach discussed in [4]. In the following we shall be working within the class of all hemirings.

A semiring $(A,+,\,.\,)$ is called a hemiring if (i) '+' is commutative (ii) there exists an element $0 \in A$ such that 0 is the identity of $(A,+,)$ and the zero element of $(A,.)$.

$$\text{i. e.}\quad 0\,a = a\,0 = 0,\ \forall\ a \in A$$

Lower radical classes for hemirings can be constructed similar to the construction of lower radicals for rings (see [3, 6, 8]).

If A is a hemiring then $HA, K_1(A)$ denote the set of all homomorphic images of A and the set of all k-semi-ideals of A respectively. If I is an k-semi-ideals of A, then we denote $I \underset{K}{\leq} A$.

First we include necessary preliminary, let ω be the universal class of all hemirings. By using ring theoretical approach discussed in [6], let $\mathcal{M}$ be a regular class of hemiring, define

$$U\,\mathcal{M} = \{\, A \in \omega : H\,A \cap \mathcal{M} = 0\}$$

then the class $U\mathcal{M}$ is a radical class and is called the upper radical class determined by the class $\mathcal{M}$. For undefined terms of hemirings we may refer (see [1, 2, 5, 6]).

2. Radical and Semisimple Classes

We extend the result of [4] by using the above construction of upper radical for hemiring which is indeed provides an excellent and different approach to handle the many results of [4] in the framework of hemiring.

The following definition is taken from S. M. Yusuf and M. Shabir [8]. The semisimple class Sρ of a radical class ρ is defined as the class of all hemirings having zero ρ-radical.

This can be rephrases in the following form.

Definition 2.1 [8]. *Let $S \subseteq \omega$, S is said to be a semisimple class, if the following two axioms are satisfied:*

S_1) $A \in S \Rightarrow HI \cap S \neq 0, \forall (0 \neq I) \in K_1(A)$

S_2) *Let $A \in \omega$ such that $HI \cap S \neq 0$, $\forall (I \neq 0)\ I \underset{K}{\leq} A$. then $A \in S$.*

A subclass of hemirings ω satisfying the condition (S_1) is called a regular class.

Definition 2.2 [8]. *Let ρ be a radical class of hemirings. Then we define a class $S\rho$ as follows:*

$$S\rho = \{ A \in \omega : \rho(A) = 0\}.$$

Theorem 2.3. *Every hereditary class is regular.*

Theorem 2.4. *$S\rho$ is hereditary.*

Theorem 2.5. *Let ρ be a radical class and $S\rho = \{ A \in \omega : \rho(A) = 0\}$. Then $S\rho$ is a semisimple class.*

Proof: *By Theorem 2.3 and Theorem 2.4, $S\rho$ is regular class i.e. (S_1) is satisfied.*

S_2) *Let $A \in \omega$ such that $HI \cap S\rho \neq 0$, $\forall (I \neq 0) \underset{K}{\leq} A$, we claim that $A \in S\rho$.*

Assume on contrary $A \in S\rho$, therefore $\rho(A) \neq 0$. Now $(\rho(A) \neq 0) \underset{K}{\leq} A$, let $I = \rho(A)$. Let $I/J \in HI \cap S\rho = H(\rho(A)) \cap S\rho$. This implies that $\rho(A)/J \in S\rho$. Since ρ is a radical class, therefore ρ *is* homomorphically closed and $\rho(A) \in \rho$, therefore $\rho(A)/J \in \rho$ and we have $\rho(\rho(A)/J) = \rho(A)/J$. As $\rho(A)/J \in S\rho$. Thus $\rho(\rho(A)/J) = 0$. This implies that $\rho(A)/J = 0$ and $\rho(A) \subseteq J$. This implies that $\rho(A) = J\ (\because J \subseteq \rho(A))$ and hence $I/J = \rho(A)/J = 0$. As I/J is an arbitrary element of $HI \cap S\rho$ such that $I/J = 0$, therefore we have $HI \cap S = 0$, for some k-semi-ideal $I = \rho(A) \neq 0$. This contradicts the fact $HI \cap S\rho \neq 0, \forall (I \neq 0) \underset{K}{\leq} A$. Consequently, $\rho(A) = 0$ and hence $A \in S\rho$ and (S_2) is satisfied.

Definition 2.6. *Let ρ_1, ρ_2 be radical classes of hemirings, then we define their sum*

$$\rho_1 + \rho_2 = \{A \in \omega : \rho_1(A) + \rho_2(A) = A\}.$$

We write $(\rho_1 + \rho_2)(A) = \rho_1(A) + \rho_2(A)$ for all $A \in \omega$.

Definition 2.7. *Let $\rho_1 + \rho_2$ be radical classes of hemirings. Then*

$$S(\rho_1 + \rho_2) = \{A \in \omega : (\rho_1 + \rho_2)(A) = 0\}.$$

We now investigate conditions under which $\rho_1 + \rho_2$ will be a radical class.

In the case of hemirings one can easily prove all the standard result concerning radical classes, sum of two radical classes, k-semi-ideal and semisimple classes. Here we mention only few of them, which can be obtained on the line of rings theoretical approach.

Theorem 2.8. *If ρ_1, ρ_2 are radical classes of hemirings, then*

$$S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$$

Theorem 2.9. *If ρ_1 and ρ_2 are radical classes of hemirings, then*

$$(\rho_1+\rho_2) \cap S(\rho_1+\rho_2) = 0.$$

Theorem 2.10. *If ρ_1 and ρ_2 are radical classes, then $\rho_1(A)+\rho_2(A)$ is the largest $\rho_1+\rho_2$ semi-ideal of the hemirings A.*

Theorem 2.11. *Let ρ_1, ρ_2 be radical classes of hemirings and $I \underset{K}{\leq} A$, then*

$$(\rho_1+\rho_2)(I) \subseteq (\rho_1+\rho_2)(A) \cap I.$$

Theorem 2.12. *Let ρ_1, ρ_2 be radical classes of hemirings, then $S(\rho_1+\rho_2)$ is a semi-simple class of hemiring.*

Proof: Let $A \in S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$ (by Theorem 2.9). This implies that $A \in S\rho_1$ and $A \in S\rho_2$ and hence $\rho_1(A) = 0$ and $\rho_2(A) = 0$. Let $I \underset{K}{\leq} A$, then by hereditary of $S\rho_1$ and $S\rho_2$, $I \in S\rho_1$ and $I \in S\rho_2$. This implies $I \in S\rho_1 \cap S\rho_2$ and hence $I \in S(\rho_1+\rho_2)$. Thus $S(\rho_1+\rho_2)$ is hereditary. This implies that if $A \in S(\rho_1+\rho_2)$ and I is a non-zero k-semi-ideal of A, then its non-zero homomorphic image in $S(\rho_1+\rho_2)$ is I itself. Therefore (S_1) of the definition 2.1 is satisfied. Suppose every non-zero k-semi-ideal of a hemiring A has a non-zero homomorphic image in $S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$. This means every non-zero k-semi-ideal of a hemiring A has a non-zero homomorphic image in $S\rho_1$ and also in $S\rho_2$. This implies $A \in S\rho_1$ and $A \in S\rho_2$ because $S\rho_1, S\rho_2$ are semisimple classes. Consequently, we have $A \in S\rho_1 \cap S\rho_2 = S(\rho_1+\rho_2)$. This shows that (S_2) of the definition 2.1 is also satisfied.

Theorem 2.13. Let ρ_1, ρ_2 *be radical classes of hemirings. Then $\rho_1+\rho_2$ is hereditary if and only if $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I, \forall A \in \omega, \forall I \underset{K}{\leq} A$.*

Proof: Let $\rho_1+\rho_2$ be hereditary and $I \underset{K}{\leq} A$. Now $(\rho_1+\rho_2)(A) \in \rho_1+\rho_2$ and $(\rho_1+\rho_2)(A) \cap I \underset{K}{\leq} (\rho_1+\rho_2)(A)$. By hereditary of $\rho_1+\rho_2$, we have $(\rho_1+\rho_2)(A) \cap I \in \rho_1+\rho_2$. As $(\rho_1+\rho_2)(A) \cap I \leq I$, therefore, we have $(\rho_1+\rho_2)(A) \cap I \subseteq (\rho_1+\rho_2)(I)$. Also we have $(\rho_1+\rho_2)(I) \subseteq (\rho_1+\rho_2)(A) \cap I$ (by theorem 2.12). Consequently, we have $(\rho_1+\rho_2)(A) \cap I = (\rho_1+\rho_2)(I)$.

Conversely, assume that $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I, \forall A \in \omega, \forall I \underset{K}{\leq} A$. Let $A \in \rho_1+\rho_2$ and $I \underset{K}{\leq} A$, then $(\rho_1+\rho_2)(A) = A$. Thus $(\rho_1+\rho_2)(I) = (\rho_1+\rho_2)(A) \cap I = A \cap I = I$. This shows that $I \in \rho_1+\rho_2$ and hence $\rho_1+\rho_2$ is hereditary.

Corollary 2.14. *Let* ρ_1 and ρ_2 *be hereditary radical classes of hemirings. Then the class* $\rho_1 + \rho_2$ *is hereditary if and only if*

$$I \cap \rho_1(A) + I \cap \rho_2(A) = I \cap (\rho_1(A) + \rho_2(A)), \forall\ A \in \omega, \forall\ I \underset{K}{\leq} A.$$

3. The upper radical of $\mathsf{S}(\rho_1+\rho_2)$

The notion of upper radical classes or upper radicals was originally introduced by F. A. Szasz ([5, see [6]) for rings.

In the case of hemirings one can easily prove all the standard result concerning lower radicals, upper radicals and regular classes. Here we mention only few of them, which can be obtained on the line of rings theoretical approach.

Theorem 3.1. *Let* $\mathcal{M}$ *be a regular class of hemirings, define* $\mathsf{U}\mathcal{M} = \{A \in \omega : HA \cap \mathcal{M} = 0\}$ *then the class* $\mathsf{U}\mathcal{M}$ *is a radical class and is called the upper radical class determined by the class* $\mathcal{M}$.

Theorem 3.2. *If* $\mathcal{M}$ *is regular class of hemirings such that* $\mathcal{M} \subseteq \mathsf{S}(\rho_1 + \rho_2)$, *then* $(\rho_1 + \rho_2) \subseteq \mathsf{U}\mathcal{M}$.

Theorem 3.3. *If* ρ_1 *and* ρ_2 *are radical classes of hemirings, then*

$$\mathsf{U}\mathsf{S}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}\rho_1 + \mathsf{S}\rho_2 = 0\}.$$

Proof: *Since* $\mathsf{S}(\rho_1 + \rho_2)$ *is a semisimple class, so by definition, we have*

$$\mathsf{U}\mathsf{S}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}(\rho_1 + \rho_2) = 0\}.$$

Since $\mathsf{S}(\rho_1 + \rho_2) = \mathsf{S}\rho_1 \cap \mathsf{S}\rho_2$ (by Theorem 2.9). Therefore we have

$$\mathsf{U}\mathsf{S}(\rho_1 + \rho_2) = \{A \in \omega : HA \cap \mathsf{S}\rho_1 + \mathsf{S}\rho_2 = 0\}.$$

Theorem 3.4. *Let* ρ_1 and ρ_2 *be radical classes of hemirings. Then*

$$\mathsf{S}(L(\rho_1 + \rho_2)) = \mathsf{S}(\rho_1 + \rho_2).$$

Proof : Let $A \in \mathsf{S}(L(\rho_1 + \rho_2))$, we will show that $A \in \mathsf{S}(\rho_1 + \rho_2)$. Assume on contrary that $A \notin \mathsf{S}(\rho_1 + \rho_2)$ but $A \in \mathsf{S}(L(\rho_1 + \rho_2))$ implies that $[L(\rho_1 + \rho_2)](A) = 0$. Since $A \notin \mathsf{S}(\rho_1 + \rho_2)$ $= \mathsf{S}\rho \cap \mathsf{S}\rho$, therefore $A \notin \mathsf{S}\rho_1 \cap \mathsf{S}\rho_2$. This implies that $A \notin \mathsf{S}\rho_1$ or $A \notin \mathsf{S}\rho_2$. Now $A \notin \mathsf{S}\rho_1$. This implies that $\rho_1(A) \neq 0$. As $0 \neq \rho_1(A) \in \rho_1 \subseteq \rho_1 \cup \rho_2 \subseteq L(\rho_1 \cup \rho_2) = L(\rho_1 + \rho_2)$. Since $A \in \mathsf{S}(L(\rho_1 + \rho_2))$ and $\mathsf{S}(L(\rho_1 + \rho_2))$ is heriditary, we have $\rho_1(A) \in \mathsf{S}(L(\rho_1 + \rho_2))$ and hence $\rho(A) \neq 0) \in \mathsf{S}(L(\rho_1 + \rho_2)) \cap (L(\rho_1 + \rho_2)$. We have a contradiction that $[L(\rho_1 + \rho_2)] \cap \mathsf{S}(L(\rho_1 + \rho_2)) = 0$. This proves that $A \in \mathsf{S}(\rho_1 + \rho_2)$. Now we have $A \in \mathsf{S}(L(\rho_1 + \rho_2))$ implies that $A \in \mathsf{S}(\rho_1 + \rho_2)$. This shows that $\mathsf{S}(L(\rho_1 + \rho_2)) \subseteq \mathsf{S}(L(\rho_1 + \rho_2))$.

For reverse inclusion, let $A \notin \mathsf{S}(L(\rho_1 + \rho_2))$. We shall show that $A \notin \mathsf{S}(\rho_1 + \rho_2)$. Assume on contrary that $A \in \mathsf{S}(\rho_1 + \rho_2)$ but $A \notin \mathsf{S}(L(\rho_1 + \rho_2))$ implies that $[L(\rho_1 + \rho_2)](A) \neq 0$. Let $I = [L(\rho_1 + \rho_2)](A)$. Then I has a non zero accessible $\rho_1 + \rho_2$ -sub-hemiring, say, T. Now

$A \in S(\rho_1+\rho_2) = S\rho_1 \cap S\rho_2$ and $I \leq_{\kappa} A$. By the heriditary of $S(\rho_1+\rho_2)$, $I \in S(\rho_1+\rho_2)$. Now T is an accessible sub-hemiring of I and so $T \in S(\rho_1+\rho_2)$. This implies that $(\rho_1+\rho_2)(T) = 0$. Since $T \in \rho_1+\rho_2$, therefore $(\rho_1+\rho_2)(T) = T$ and hence $T = 0$, which contradicts that $T \neq 0$. Thus $S(\rho_1+\rho_2) \subseteq S(L(\rho_1+\rho_2))$. Consequently, we have $S(L(\rho_1+\rho_2)) = S(\rho_1+\rho_2)$. This completes the proof.

Corollary 3.5. *If ρ_1, ρ_2 are radical classes of hemirings, then*

$$L(\rho_1 \cup \rho_2) = \mathsf{U}[S(\rho_1+\rho_2)]$$

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