The Nepali Math. Sc. Report
Vol. 21 No. 1&2, 2003

A Simplified Model For General Circulation In Earth's Atmosphere And Ocean

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Abstract : In order to understand the behavior of the atmosphere and the ocean, a various types of mathematical models governing the motion and the states of atmosphere and ocean have been already established and studied by different mathematicians. Present paper is devoted to the construction of a simplified linear model that governs the general circulation of atmosphere and ocean.

1. Introduction

Mathematical modeling is what physical applied mathematics is all about. A model is a representation of a process. Usually, a mathematical model takes the form of a set of equations describing a number of variables.

Mathematical modeling begins with the identification of a problem. Once a problem is identified, and a mechanism proposed, then one must formulate it mathematically. Formulation involves equations and boundary conditions. Physical laws governing the motion and states of atmosphere and ocean can be described by the general equations of hydrodynamics and thermodynamics, which are very complicated.

There are essentially two characteristics of both the atmosphere and ocean that are used in simplifying the equations. The first one is that for large-scale geophysical flows, the ratio between the vertical and horizontal scales is very small. Another small parameter is the ratio of the speed (horizontal) of wind to the speed of rotation of the earth around the polar axis. This number, called the Rossby number, is of order of 1/50. The asymptotics corresponding to the small Rossby number leads to the so-called geostrophic and quasi-geostrophic equations.

Reduction is the process whereby a model is simplified, most often by neglecting various small terms. But, what's meant by "small"? For example ; a speed of 1 cm per second is slow for a bullet, but fast for an earthworm. So, the word 'small' is to be used in a relative manner.

Linerization is a type of reduction, which is basic to practically all-analytic models of atmospheric motions. In this process, the flow field is divided into a longitudinally averaged part (zonal mean) and deviations from that average (perturbation). It is then assumed that the perturbations are sufficiently small so that the terms involving products of the perturbations may be neglected compared to linear terms.

It is well known that in order to understand the turbulent behavior of the atmosphere and the ocean, and to predict the weather and the climate, we need to establish some mathematical equations or models governing the motion and the states of the atmosphere and the ocean. We also need to establish and solve the corresponding numerical models (the numerical approximation of the mathematical equations).

The aim of the author is to derive some mathematical models for the coupling of the atmosphere and the ocean to study them from analytical as well as numerical viewpoints. As a first step, the present paper is devoted to the construction of a simplified linear mathematical model that governs the general circulation of atmosphere and ocean. Here in this paper, a very simple global circulation model of the atmosphere and the ocean is obtained, for which the equations of motion for wind and temperature are linear evolution equations similar to the linear Stokes equations.

2. Historical Background

As astronomers discover new planets, the planets are named after them. Similarly, scientists name the equations after the names of those mathematicians, who develop them. There are many equations named after mathematicians such as Laplace's equation, Euler-D 'Alambert's equation, Tricomi's equation, Schrodinger equation, Maxwell's equation and so on.

Euler derived the equations that describe the most fundamental behavior of a fluid in 1755. These are the equations of conservation of momentum and conservation of mass of a fluid that is incompressible and is inviscid. The initial-boundary value problem for the Euler equations is a surprisingly difficult problem. Perhaps, it is one of the most challenging of all problems in partial differential equations that arise directly from physics. Even the basic questions of existence and uniqueness of the solutions in three dimensions remain open.

All real fluids are at least very weakly viscous. Viscosity is necessary to generate flows, and its influence is very complicated. Incorporation of the effects of viscosity leads to the versions of Euler equations, called Navier–Stokes equations. Since friction is a fact of nature, it could be argued that only the Navier–Stokes equations are physically relevant. But there is much to learn from the Euler equations.

The issue of the stability or instability of a fluid flow became one of the most basic problems in fluid dynamics and was examined experimentally and mathematically by such giants of science as Helmholtz, Kelvin, Rayleigh and Reynolds. An elegant mathematical treatment of non-linear stability is given by Arnold [2], which is applicable to certain two dimensional inviscid fluid motions.

The idea of wide application of mathematical models of rotating fluids to the study of dynamics of atmospheric processes belongs to A. Friedman, who in the beginning of 20th century contributed a series of fundamental works in this direction.

The linearized Navier-Stokes system

$$(2.1)\qquad \begin{cases} \dfrac{\partial \vec{V}}{\partial t} - [\vec{v},\vec{\omega}] - \nu\Delta\vec{v} + \nabla P = \vec{F} \\ \operatorname{div} \vec{V} = 0 \end{cases}$$

describing the motion of a rotating fluid was studied by V.N. Maslennikova [8]. System (2.1) without the Coriolis term $[\vec{v},\vec{\omega}]$ was extensively studied from the mathematical point of view by Oseen and Laray. In particular, Oseen was the first to construct the fundamental solution of system (2.1) without taking rotation into account, the so-called *Oseen tensor*. System (2.1) without viscosity. (i.e., $\upsilon = 0$) was first studied by S.L. Sobolov and is known as *Sobolov system*. Even in a system with viscosity, as well as in a Sovolev system [1], the presence of the terms $[\vec{v},\vec{\omega}]$ causes the solution to have an oscillatory character and leads to the emergence of a vortex, whereas in the absence of these terms a solution of system (2.1) is in a definite sense analogous to a solution of the heat equation.

L. Marchhuk introduced a linearized system [7] of partial differential equations in his book 'Mathematical Models of Circulation in Oceans' in 1980, in which a numerical approach is suggested for its solution.

3. Formulation of the Mathematical Model

Let the whole of the space $\mathbb{R}^3$ be filled with inviscid fluid being at rest at infinity. Let *Ox* be an inertial frame of reference and *Oy* be a frame of reference which rotates with respect to *Ox* with constant angular velocity $\vec{\omega}$. Without loss of generality, we assume

$$\vec{\omega} = (0, 0, \omega)$$

The dynamics of an inviscid fluid with respect to the inertial frame Ox is governed by the Euler equations

$$(3.1)\qquad \begin{cases} \dfrac{\partial \vec{V}}{\partial t} + (\vec{V}, \nabla_x)\vec{V} + \dfrac{1}{\xi}\nabla_x P = \vec{F}(x,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_x(\xi \vec{v}) = 0, \end{cases}$$

where

$\vec{V}$ = fluid's velocity field
P = hydrodynamics pressure
ξ = fluid's density
$\vec{F}$ = mass density of external forces

With respect to the rotating frame Oy, the dynamics of inviscid fluid is known to be governed [5] by the equations

$$(3.2)\qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla_y)\vec{u} - 2[\vec{u}, \vec{\omega}] + \dfrac{1}{\xi}\nabla_y Q = \vec{G}(y,t) \\ \dfrac{\partial \xi}{\partial t} + \operatorname{div}_y(\xi \vec{u}) = 0. \end{cases}$$

with $\vec{u}, Q$ and $\vec{G}$ conveying the same physical meaning as $\vec{V}$, P and $\vec{F}$ respectively.

In equations (3.2), the pressure Q involves the centrifugal force :

$$Q = P - \frac{\xi}{2}|[\vec{y}, \vec{\omega}]|^2$$

and the velocity $\vec{u} = \vec{v} - [\vec{\omega}, \vec{y}]$. Since the fluid is at rest at infinity with respect to Ox, we have :

$$(3.3)\qquad \lim_{|x|\to\infty} \vec{V}(x,t) = 0.$$

With respect to Oy, the fluid is not at rest at infinity :

$$(3.4)\qquad \lim_{|y|\to\infty} (\vec{u} + [\vec{\omega}, y]) = 0.$$

Note that in (3.1), the pressure $P(x,t)$ is bounded at infinity. But in (3.2), the pressure $Q(y,t)$ at infinity becomes a polynomial in y of the second order. And due to condition (3.4), the velocity field $u(y,t)$ in (3.2) becomes at infinity a polynomial in y of the first order.

Technically, the condition (3.4) makes solving any problem for equations (3.2) most complicated, compared to the case of zero condition at infinity. To overcome this technical obstacle, we represent the solution $\vec{u}$ of (3.2) in the form

$$\vec{w}(y,t) = \vec{u}(y,t) + [\vec{\omega}, \vec{y}] \tag{3.5}$$

which reduces equations (3.2) to the following form:

$$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} + (\vec{w}, \nabla_y)\vec{w} - 2[\vec{w}, \vec{\omega}] + \dfrac{1}{\xi} \nabla_y p = \vec{H}(y,t) \\ \dfrac{\partial \xi}{\partial t} + \text{div}_y (\xi \vec{w}) = 0, \end{cases} \tag{3.6}$$

where

$$\vec{w}(y,t) = \vec{u}(y,t) + [\vec{\omega}, \vec{y}] = \vec{v}(S^* y, t)$$

$$p(y, t) = P(S^* y, t)$$

with S being a rotation matrix of the following form

$$S = S(t) = \begin{pmatrix} \text{Cos}\, \omega t & -\text{Sin}\, \omega t & 0 \\ \text{Sin}\, \omega t & \text{Cos}\, \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that the equation (3.6) could be derived from equation (3.1) by the change of variables

$$y = S(t)\, x,$$

where S is an orthogonal matrix :

$$S^{-1} = S^*$$

In equations (3.6), the pressure $p(y, t)$ does not involve the centrifugal force, hence $p(y, t)$ is bounded at infinity as $P(x, t)$, with the velocity $\vec{w} = \vec{w}(y,t)$ being in fact a velocity of the fluid with respect to Ox, but measured in Oy. And only at this point one can linearize equations (3.6) for sufficiently small velocities :

$$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} - 2[\vec{w}, \vec{\omega}] + \dfrac{1}{\xi} \nabla_y p = H(y,t) \\ \dfrac{\partial \xi}{\partial t} + \text{div}_y (\xi \vec{w}) = 0. \end{cases} \tag{3.7}$$

The equations (3.7) of dynamics are supplemented by the equation of heat transfer for continuous medium :

$$\frac{\partial T}{\partial t} + (\vec{w}, \nabla_y T) = f(y,t), \tag{3.8}$$

where

T is the temperature,

f is the heat source density.

The difference in temperature at different points creates a force acting upon the particles of fluid. One of the simplest models of such force [4] is

$$\vec{H}(y,t) = \vec{h}(y,t) - \vec{g}\left(\frac{T - T_0}{T_0}\right), \tag{3.9}$$

where

T_0 is some standard temperature,
$T - T_0$ is the deviation of the temperature T from T_0 ,
$\vec{h}(y,t)$ is the mass density of external forces,
$\vec{g}$ is the vector of free fall acceleration

Equation (3.9) describes the free heat convection in the Earth's atmosphere. We consider the simple case of vectors $\vec{g}$ and $\vec{\omega}$ being collinear.

Now, we restrict ourselves to the case of incompressible fluid, and without loss of generality, we assume $\xi = 1$. So, equations (3.7) – (3.9) take the form :

$$\begin{cases} \dfrac{\partial \vec{w}}{\partial t} - 2\,[\vec{w},\vec{\omega}] + \nabla_y p + \dfrac{T'}{T_0}\,\vec{g} = \vec{h}(y,t) \\ \operatorname{div} \vec{w} = 0 \\ \dfrac{\partial T'}{\partial t} + (\vec{w}, \nabla_y T') = f(y,t), \end{cases} \tag{3.10}$$

where $T' = T - T_0$ Equations (3.10) represent a simplified model for general circulation in Earth's atmosphere and Oceans [6, 7, 10]. Marchhuk and others were the first to introduce and study the equations (3.10). Another version of simplified equations may be found in the recent works of Lion, Temam and Wang [6].

Substituting $\vec{V}, P, T, \vec{\omega}$ and $\vec{F}$ for $\vec{w}, p, T', 2\vec{\omega}$ and $\vec{h}$ respectively, we get

$$\begin{cases} \dfrac{\partial \vec{V}}{\partial t} - [\vec{V},\vec{\omega}] + \nabla p + \vec{g}\left(\dfrac{T}{T_0}\right) = \vec{F} \\ \operatorname{div} \vec{V} = 0 \\ \dfrac{\partial T}{\partial t} + (\vec{V}, \nabla T) = f \end{cases} \tag{3.10$'$}$$

Recall that the vector product

$$[\vec{v},\vec{\omega}] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V_1 & V_2 & V_3 \\ 0 & 0 & \omega \end{vmatrix} = \omega V_2 \vec{i} - \omega V_1 \vec{j} + 0\vec{k}$$

Let $\vec{V} = (\vec{V}_1, \vec{V}_2, \vec{V}_3)$, $\vec{F} = (F_1, F_2, F_3)$ and $\frac{g}{T_0} = \sigma$, then the dynamics equations in $(3.10')$ can be rewritten as follows :

$$\text{(3.11)} \quad \begin{cases} \dfrac{\partial V_1}{\partial t} - \vec{\omega} V_2 + \dfrac{\partial P}{\partial x_1} = F_1 \\ \dfrac{\partial V_2}{\partial t} + \vec{\omega} V_1 + \dfrac{\partial P}{\partial x_2} = F_2 \\ \dfrac{\partial V_3}{\partial t} + \dfrac{\partial P}{\partial x_3} + \sigma T = F_3 \end{cases}$$

Taking into account only the vertical gradients of temperature, we find

$$(\vec{V}, \nabla T) = V_1 \frac{\partial T}{\partial x_1} + V_2 \frac{\partial T}{\partial x_2} + V_3 \frac{\partial T}{\partial x_3}$$

$$= V_3 \frac{\partial T}{\partial x_3}, \text{ taking } \frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$$

$$= -r V_3 \ (r > 0),$$

where $r = -\frac{\partial T}{\partial x_3}$ is being treated as a prescribed constant. The heat transfer equation in $(3.10')$ now reduces to the form

$$\text{(3.12)} \quad \frac{\partial T}{\partial t} - r V_3 = f.$$

Thus, we have derived the linear system of partial differential equations

$$\text{(3.13)} \quad \begin{cases} \dfrac{\partial V_1}{\partial t} - \omega V_2 + \dfrac{\partial P}{\partial x_1} = F_1 \\ \dfrac{\partial V_2}{\partial t} + \omega V_1 + \dfrac{\partial P}{\partial x_2} = F_2 \\ \dfrac{\partial V_3}{\partial t} + \sigma T + \dfrac{\partial P}{\partial x_3} = F_3 \\ \dfrac{\partial T}{\partial t} - r V_3 = f \\ \operatorname{div} \vec{V} = 0. \end{cases}$$

System (3.13) represents a simplified linear model for general circulation of the atmosphere and ocean [1].

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