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A Topological Criterion for Starlikeness, Piecewise Convex and Piecewise α - Convex Functions

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Abstract: In [3] C.N. Genther, St Ruscheweyh and L. C. Salinas introduced the concept of quasi - simple curve and have given a criterion for it. In this article we shall use the concept of quasi - simple curves to establish a topological criterion for starlike, piecewise convex and piecewise α - convex functions.

1. Introduction

Definition 1.1 Let C denote the entire complx plan. A positively oriented close curve γ is siad to be quasi - simple if it is the positively oriented boundary of a simply connected domain. An arc is said be a quasi - simple if it is a subarc of a quasi- simple curve. For any closed curve or arc $\gamma : [a\ b] \longrightarrow C$, let $S_{\gamma} := \{\gamma(t) : a \leq t \leq b\}$, be its support in C.

Definition 1.2 Let γ be a positively oriented closed curve and ω_0 be any point in the complex plane C. We say that ω_0 is attainable with respect to γ from ∞ if there exist simply connected domain G such that $0 \in \partial G$, the function $f(z) = z^2 + \omega_0$ is univalent in G, and there exists a closed curve γ^* such that $S_{\gamma^*} \subseteq \overline{G}$ and $f(\gamma^*(t)) = \gamma(t)$; $t \in [0\ 2\pi]$.

In a More descriptive language we can say that ω_0 is attainnable with respect to γ from ∞ if there exists a curve connecting ω_0 with ∞ which does not intersect the curve γ (it may, however, touch γ).

Let A_γ denote the set of all attainable points with respect to γ from ∞. Clearly A_γ is the union of the closure of some of the connected components of S_γ^c, the complement of S_γ, including the unbounded component plus possibly certain segment of S_γ. Let D_γ is the simply connected domain bounded by the quasi - simple curve γ. Then $A_\gamma = D_\gamma^c$ and in particular

$$\gamma \text{ is quasi - simple} \Longrightarrow S_\gamma \subseteq A_\gamma \qquad (1)$$

Definition 1.3 An oriented closed curve $\gamma : [0\ 2\pi] \longrightarrow C$ is said to be in the class **C** if it has the following properties :

C$_1$: γ is piecewise smooth.

C$_2$: γ is locally quasi - simple i.e. for each $t \in [\ 0\ 2\pi]$ there exists $\varepsilon(t) > 0$ such that the arc $\gamma_t := \gamma/[t - \in\ \ t + \in]$ is quasi - simple.

C$_3$: for every $t \in [\ 0\ 2\pi)$ let G_t be a simply connected domain which has γ_t in its (positively oriented) boundary. Then there exists an open neighbourhood U of $z_t = \gamma(t)$ for which $(U \cup G_t) \cup A_\gamma = \Phi$.

C$_4$: the function $\beta_t = \lim\limits_{\tau \to t^+} \arg \dot{\gamma}(\tau), t \in R$ satisfies $\beta(t + 2\pi) - \beta(t) = 2\pi, t \in R$

2. Statement and Proof of the Main Result

Theorem 2.1[1]

Let f be a function holomorphic in the closed unit disk $\bar{D}$, except possibly at a finite number of points in ∂D, and continuous throughout $\bar{D}$, normalized by

1 this theorem has been proved in [1], but here we have given a completely different and very short proof, as compared to the proof given in [1], by using the concept of quasi – simple curves introduced and defined in [3]

$f(0) = 0, f'(0) = 1$. Let f be locally univalent on D, $f(z) \neq 0$ on ∂D, the curve $\gamma(\theta) = f(e^{i\theta}) \in$ and

$$Re\left(1 + \frac{z f''}{f'}\right) \geq 0 \text{ on } \partial D \tag{2}$$

except on the set $\mathcal{M} = \{z \in \partial D : f'(z) = 0 \text{ or } f \text{ is not holomorphic at } z\}$.

Let the values of θ with $e^{i\theta} \in$ M be $\theta_0, \theta_1, \theta_2 \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n = \theta_0 + 2\pi$. Furthermore, let for any such point $e^{i\theta j}$, if we measure the argument of the tangent to the curve $\{f(e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$ from a point $f(e^{i\tilde{\theta}})$ where $< \theta_j$ and $\tilde{\theta}$ close to θ_j, there exist $k_j \in N$, $\alpha_j, \beta_j \in [0\ \pi)$ such that

$$\lim_{\theta \to \theta_j^+} \arg\left[ie^{i\theta} f'(e^{i\theta})\right] = \arg f(e^{i\theta}) + 2\pi k_j + \alpha_j \tag{3}$$

$$\lim_{\theta \to \theta_j^-} \arg\left[ie^{i\theta} f'(e^{i\theta})\right] = \arg f(e^{i\theta}) + (2\pi k_j + 1)\pi - \beta_j \tag{4}$$

then f is starlike.

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$ we first show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi -simple and the points $f(e^{i\theta})$ are attainable with respect to γ from ∞ which guarantees the univalence of the function f on D [3]

If $\mathcal{M}$ has no element then we are back in the classical case and the function f is in fact convex and hence starlike so nothing has to be proved. Now suppose that $\mathcal{M}$ has at least one element. It is easily seen that the quasi - simple property of the arc γ_j can not be destroyed by a negative loop, since this would mean that arg of the tangent at γ_j decreases over a certain interval but this has been ruled out by (2). The other way to destroy the quasi - simple property, namely a positive loop, can not occur either as one can readily verifies by using the construction in [3] that maps the situation on to D preserving the loops (and their orientation). So we see that any positive loop on one of γ_j would increase the total tangent rotation by 2π, but there are no negative loops available to compensate for that. Since $\gamma \in \mathbf{C}$ and hence condition C_4 limits this total rotation to the minimal value of 2π, so there is no room for positive loops, and the γ_j must be quasi - simple.

To show that the points $f(e^{i\theta})$ are attainable with respect to $\gamma = f(e^{i\theta})$ from the ∞ we first note that (3) and (4) implies that there exists $\delta > 0$ such that arg $f(e^{i\theta})$ increases on the intervals $[\theta_{j+1} - \delta, \theta_j)$ and $[\theta_j, \theta_j + \delta)$ for $j = 0, 1, 2, \ldots, n$. See [1] for detail. And we already have shown that there is no rooms for loops so it is clear that the straight line emanating from the point $f(e^{i\theta_j})$ goes to ∞ without intersecting the curve γ. Hence the point $f(e^{i\theta_j})$ are attainable with respect to γ from the ∞. Therefore the curve $\gamma = f(e^{i\theta})$ is quasi – simple and hence [3] the function f is univalent.

To show that f is starlike it remains to show that age $f(e^{i\theta})$ is increasing and the total change in argument of $f(e^{i\theta})$ as θ varies form 0 to 2π is 2π. The last assertion is clear because $f(0) = 0$, f is univalent on D and f has no zeros on ∂D, so by argument principle $\Delta_{C(=e^{i\theta})}\, f(e^{i\theta}) = 2\pi$.

To show that arg $f(e^{i\theta})$ is increasing it suffices to show that it is increasing on $[\theta_0, \theta_1]$. We know already that arg $f(e^{i\theta})$ is increasing on $[\theta_0, \theta_0 + \delta]$ and on $[\theta_1 - \delta, \theta_1]$ for some $\delta > 0$. Hence it remains to show that arg $f(e^{i\theta})$ is increasing on $[\theta_0 + \delta, \theta_1 - \delta]$. But if are $f(e^{i\theta})$ is strictly decreasing on certain interval of $[\theta_0 + \delta, \theta_1 - \delta]$ then either there should be at least one loop (positive) on γ or arguments of the tangent should decrease over certain subinterval of $\gamma_1 = f(e^{i\theta})$, $\theta_0 \leq \theta \leq \theta_1$. But both possibilities are ruled out so are $f(e^{i\theta})$ is increasing for $\theta_0 \leq \theta \leq \theta_1$ and hence on whole γ. This completes the proof.

3. Topological Criterion For Piecewise Convex And Piecewise α-Convex Functions

Definition 3.1 A quasi – simple curve γ in C is said to be n – piecewise convex curve if there are n points $z_k = \gamma(t_k)$ on S_γ with $t_1 < t_2 < \ldots < t_n < t_{n+1} = t_1 + 2\pi$ such that the function

$$\beta(t) = \lim_{\tau \to t^+} \arg \dot{\gamma}(\tau)$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$. In other words the quasi – simple curve γ is n – piecewise convex if

$$Re\left(1 + \frac{z f''}{f'}\right) \geq 0$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$ and the corresponding normalized univalent function f which maps the unit disk D conformally on to the simply connected domain D_γ is called the n-piecewise convex function.

Definition 3.2 Let γ be a quasi – simple curve in C and $\omega \notin S_\gamma$. Then the curve γ is said to be n piecewise α - convex $(0 \leq \alpha \leq 1)$ curve if there are n-points $z_k = \gamma(t_k)$ on S_γ with $t_1 < t_2 < \ldots t_n < t_{n+1} := t_1 + 2\pi$ such that

$$(1 - \alpha)\arg(\gamma(t - \omega_0) + \alpha\beta(t)$$

is increasing on $S_{\gamma k} = \gamma([t_k, t_{k+1}])$, where

$$\beta(t) = \lim_{\tau \to t^+} \arg \gamma(t)$$

and the corresponding normalized univalent function f which maps the unit disk D conformally on to the simply connected domain D_γ is called the n - piecewise α - convex function.

Theorem 3.1 Let f be a function holomorphic in $\bar{D}$, except possibly at a finite number of points on ∂D, and continuous throughout $\bar{D}$, with the normalization $f(0) = 0, f'(0) = 1$. Suppose f is locally univalent on D, $f(z) \neq 0$ on ∂D, and the curve $\gamma(\theta) = f(e^{i\theta}) \in C$. Let

$$\mathcal{M} = \{z \in \partial D : f'(z) = 0 \text{ or f is not holomorphic at } z\}.$$

and the values of θ such that $e^{i\theta} \in \mathcal{M}$ be $\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \ldots < \theta_{n-1} < \theta_n := \theta_0 + 2\pi$. Then the curve $\gamma = f(e^{i\theta})$ is n-piecewise convex curve and the function f is n-piecewise convex function if

$$Re\left(1 + \frac{zf''}{f'}\right) \geq 0 \text{ on } \partial D \setminus \boldsymbol{M}. \qquad (5)$$

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in$ **C**. Therefore from [3], to prove the theorem it suffices to show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi-simple and the points $z_j = f(e^{i\theta_j})$ are attainable with respect to γ from ∞.

Suppose $\mathcal{M}$ has n elements. If $n = 1$ then the result is obvious. Now suppose $n > 1$. From (5) it is clear that neither any single arc γ_k nor any arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$ can offer a negative loop otherwise the function $\beta(t)$ is strictly decreasing on certain subinterval of either γ_j or γ_k; which contradicts the condition (5). Hence there is no room for negative loop on the whole curve γ at all. The other possibility to destroy the quasi-simple property of the arc γ_k is by forming a positive loop on it. But any positive loop on γ_k would increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. As $\gamma \in C$, so the condition C_4 limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc γ_k. Hence each arc γ_k, $k = 1, 2, \ldots, n$ are quasi - simple.

It remains to show that the points $f(e^{i\theta})$ are attainable with respect to γ from the ∞. But by using the more or less same arguments we can prove it. We already know that there is no room for negative loop on whole γ so the only possibility to destroy the attainable property of the points $f(e^{i\alpha_k})$ is by a positive loop formed on an arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$. But again any positive loop on $(\gamma_k \cup \gamma_j)$ would also increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. And since $\gamma \in \mathbf{C}$ so the condition C_4 again limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc $(\gamma_k \cup \gamma_j)$, with $j \neq k$ too. Therefore the points $f(e^{i\alpha_k})$, for $k = 1, 2, \ldots, n$ are all attainable with respect the curve γ from ∞. Hence by definition γ is a n-piecewise convex curve and the function f is n-piecewise convex function.

By using more or less same techniques and arguments one can easily proved the following theorem on n-piecewise α.

Theorem 3.2 Let f be a function holomorphic in $\bar{\mathrm{D}}$, except possibly at a finite number of points on $\partial\mathrm{D}$, and continuous throughout $\bar{\mathrm{D}}$, with the normalization $f(0) = 0, f'(0) = 1$. Suppose f is locally univalent on D, $f(z) \neq 0$ on $\partial\mathrm{D}$, the curve $\gamma(\theta) = f(e^{i\theta}) \in \mathbf{C}$. Let

$$\mathcal{M} = \{z \in \partial D : f'(z) = 0 \text{ or } f \text{ is not holomorphic at } z\}.$$

and the values of θ such that $e^{i\theta} \in \mathcal{M}$ be $\theta_0, \theta_1, \ldots, \theta_{n-1}, \theta_n$, where $\theta_0 < \theta_1 < \ldots \theta_{n-1} < \theta_n := \theta_0 + 2\pi$. Then the curve $\gamma = f(e^{i\theta})$ is n-piecewise α-convex curve and the function f is n-piecewise α-convex function if

$$(1 - \alpha)\, Re\left(1 + \frac{zf'}{f(z)}\right) + \alpha Re\left(1 + \frac{zf''}{f'}\right) \geq 0 \text{ on } \partial D \setminus \mathcal{M} \qquad (6)$$

Proof: Since the curve $\gamma(\theta) = f(e^{i\theta}) \in C$. Therefore from [3], to prove the theorem it suffices to show that each arc $\gamma_j = f(e^{i\theta})$, $\theta_j \leq \theta \leq \theta_{j+1}$ is quasi – simple and the points $z_j = f(e^{i\theta_j})$ are attainable with respect to γ from ∞. Suppose $\mathcal{M}$ has n element. If $n = 1$ then the result is obvious. Now suppose $n > 1$. From (6) it is clear that neither any single arc γ_k nor any arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$ can offer a negative loop otherwise the function $k(t)$ is strictly decreasing on certain subinterval of either γ_j or γ_k; which contradicts the condition (6). Hence there is no room for negative loop on the whole curve γ at all. The other possibility to destroy the q-s property of the arc γ_k is by forming a positive loop on it. But any positive loop on γ_k would increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that. And condition C_4 limits this total rotation to the minimal value of 2π. Therefore there is no room for a positive loop on the arc γ_k. Hence each arc γ_k, $k = 1, 2, \ldots, n$ are

q-s.

It remains to show that the points $f(e^{i\theta_k})$ are attainable with respect to γ from the ∞. But by using the more or less same arguments we can prove it. We already know that there is no room for negative loop on whole γ so the only possibility to destroy the attainable property of the points $f(e^{i\theta_k})$ is by a positive loop formed on an arc of the form $(\gamma_k \cup \gamma_j)$, the union of any two arcs γ_k and γ_j, with $j \neq k$. But again any positive loop on $(\gamma_k \cup \gamma_j)$ would also increase the total tangent rotation by 2π, but there are no negative loop available to compensate for that . And condition C_4 again limits this total rotation to the minimal value of 2π. Therefore there is no room for a posistive loop on the arc $(\gamma_k \cup \gamma_j)$, with $j \neq k$ too. Therefore the points $f(e^{i\theta_k})$, for $k = 1, 2, \ldots, n$ are all attainable with respect the curve γ from ∞. Hence by definition γ is a n-piecewise α- convex curve and the function f is n-piecewise α convex function.

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