

AN ANALYSIS OF MICROBIAL POPULATION OF CHEMOSTAT MODEL IN FUZZY ENVIRONMENT

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Abstract: Chemostat is a continuous stirred tank reactor used for continuous microbial biomass production in commercial, medical and other research problems. While modeling real world phenomena through differential equations as backbone of practical problems, we need to introduce various parameters. These parameters may be vague, imprecise and uncertain. To incorporate these uncertainties, the notion of fuzzy differential equations is used in chemostat model as one of the tool. In this paper, we discuss some new results for the stability analysis of chemostat model and the results so obtained are justifiable analytically and verified graphically in fuzzy environment.

Key Words: Chemostat, Fuzzy sets, stability theory, fuzzy differential equations, gH-differentiability.

AMS (MOS) Subject Classification. 34A07 Fuzzy Differential Equations.

1. INTRODUCTION

Applied analysis plays an important role to model real natural phenomena;like in the field of economics, biomathematics, science and engineering. But in the modern era, it is believed that the information about the physical phenomena may be uncertain. The concept of fuzzy set theory is a powerful tool to overcome these impreciseness. The concept of fuzzy set theory and differential equations separately lead to focus their work towards the new concept of the Fuzzy Differential Equations(FDE). In recent years, the research on FDE has been growing rapidly.

L. A. Zadeh[8] introduced the term fuzzy for the first time in 1965 and gave some examples where the nature of uncertainty in behavior of given system possesses a fuzzy phenomena. In 1972,the fuzzy derivative was first conceptualized by Chang and Zadeh[9], whereas the fuzzy derivative was first introduced by A. Kandel and W. J. Byatt [10], in 1978.

Based on Zadeh's extension principle[8], Dubois and Prade[13] in 1982, presented the elementary fuzzy calculus. Also, O. Kaleva [16] [17] in 1987 & 1990 and S. Sekkala [18] in 1987, studied the FDE and fuzzy initial value problem. B. Bede[11] in 2007 presented two characterization theorems for the solution of FDE which translate FDE into crisp ordinary differential equations.

In 2012, the new concept of generalized derivative for fuzzy mappings and FDEs were discussed by A. V. Platnikov and N. V. Skripnik [12]. Also, S. P. Mondal et.al [3] solved the

first order linear homogeneous ordinary differential equation in fuzzy environment in 2013. In the year 2014, N. V. Hoa and N. D. Phu [2] studied two different types of solutions of fuzzy functional integro-differential equations. In 2016, S. Paul et.al [14] described the solution of fuzzy quota harvesting model in fuzzy environment.

The logistic growth model proposed by Pierre-Francois Verhulst in 1838 is as follows:

$$(1.1) \quad \frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right)$$

where $P(t)$ represents the population size at time t , r is the intrinsic growth rate and K is the carrying capacity.

If $f(P) = -cP$, for $c > 0$, is the emigration function. Then, the modified logistic growth equation will turn out to be chemostat model. The following is the real application of the logistic growth equation with harvesting function $f(P) = -cP$ to microbial population growth in chemostat chamber. If $f(P) = -cP$, then the model equation (1.1) becomes

$$(1.2) \quad \frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - cP$$

which is microbial population growth model and this is about a schematics of a chemostat with a stock of nutrient C_0 pumped into the chamber of a bacterial culture. We assume that the chemostat chamber is well stirred so that the nutrient concentration is constant at each time t .

In this paper, we have extended this model equation for the stability analysis of the equilibrium points for fuzzy parameters when the immigration function $f(P)$ is proportional to P and we have done similar observation in the case of logistic growth model with immigration function $f(P) = cP$, for $c > 0$ in fuzzy environment [6].

2. PRELIMINARY CONCEPTS

We start with the following definitions.

Definition 2.1. [5] A fuzzy number is a function such as $u : \mathbb{R} \rightarrow [0, 1]$ satisfying the following properties:

- (1) u is normal, i.e $\exists x_0 \in \mathbb{R}$ with $u(x_0) = 1$.
- (2) u is a convex fuzzy set i.e $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \forall x, y \in \mathbb{R}, \lambda \in [0, 1]$
- (3) u is upper semi-continuous on \mathbb{R} .
- (4) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact where \overline{A} denotes the closure of A .

Definition 2.2. [5] A fuzzy number \tilde{A} is said to be triangular if its membership function $\mu_{\tilde{A}}$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } x \geq c \end{cases}$$

Definition 2.3. [1] A fuzzy number \tilde{u} in parametric form is a pair (\underline{u}, \bar{u}) of functions $\underline{u}(\alpha), \bar{u}(\alpha), 0 \leq \alpha \leq 1$ which satisfy the following conditions:

- (1) $\underline{u}(\alpha)$ is bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0.
- (2) $\bar{u}(\alpha)$ is bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0.
- (3) $\underline{u}(\alpha) \leq \bar{u}(\alpha), 0 \leq \alpha \leq 1$.

Definition 2.4. The generalized Hukuhara difference (gH-difference) [7] of two fuzzy numbers $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$ is defined as follows:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \text{ is equivalent to } \begin{cases} \text{(i)} & \tilde{u} = \tilde{v} \oplus \tilde{w} \\ \text{(ii)} & \tilde{v} = \tilde{u} \oplus (-1)\tilde{w} \end{cases}$$

In terms of α -cut $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w}$ is equivalent to $[\tilde{w}]_\alpha = [\underline{w}(\alpha), \bar{w}(\alpha)]$ where

$$\underline{w}(\alpha) = \min\{\underline{u}(\alpha) - \underline{v}(\alpha), \bar{u}(\alpha) - \bar{v}(\alpha)\}$$

and

$$\bar{w}(\alpha) = \max\{\underline{u}(\alpha) - \underline{v}(\alpha), \bar{u}(\alpha) - \bar{v}(\alpha)\}.$$

Definition 2.5. [4] The generalized Hukuhara derivative of a fuzzy valued function $f : (a, b) \rightarrow \mathcal{F}(\mathbb{R})$ at point t_0 is defined as

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}$$

If $f'(t_0) \in \mathcal{F}(\mathbb{R})$, satisfying the definition of fuzzy set, then f is generalized Hukuhara differentiable at t_0 . Also, $f(t)$ is (i) – gH differentiable at t_0 if

$$[f'(t_0)]_\alpha = [\underline{f}'(t_0, \alpha), \bar{f}'(t_0, \alpha)]$$

and $f(t)$ is (ii) – gH differentiable at t_0 if

$$[f'(t_0)]_\alpha = [\bar{f}'(t_0, \alpha), \underline{f}'(t_0, \alpha)].$$

Definition 2.6. [4] If the solution of the fuzzy differential equation of the form $[\underline{x}(t, \alpha), \bar{x}(t, \alpha)]$, is called strong solution if

$$\frac{\underline{x}(t, \alpha)}{d\alpha} > 0, \frac{\bar{x}(t, \alpha)}{d\alpha} < 0, \text{ for all } \alpha \in [0, \omega], \underline{x}(t, \omega) \leq \bar{x}(t, \omega).$$

Otherwise, it is weak solution.

3. MAIN RESULTS

Let us consider the chemostat model [19] for the microbial population growth in continuous stir tank vessel of the form

$$(3.1) \quad \frac{dP}{dt} = rP \left(1 - \frac{P}{K} \right) - cP$$

where,

$$\begin{aligned}
 P &= \text{Microbial population in the chamber at time } t \\
 r &= \text{Growth rate of the bacteria in the chamber} \\
 c &= \text{Rate of bacterial outflow from the chamber} \\
 K &= \text{Carrying capacity} \\
 \frac{dP}{dt} &= \text{Rate of change of bacterial population in the chamber.}
 \end{aligned}$$

Now, if P^* is the equilibrium point, then we get $P^* = 0$ and $P^* = K \left(1 - \frac{c}{r}\right)$.
If

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) - cP = G(P)$$

then,

$$G'(P) = r - \frac{2Pr}{K} - c.$$

At $P^* = 0$,

$$G'(0) = r - c$$

and at $P^* = K \left(1 - \frac{c}{r}\right)$.

$$G'\left(K \left(1 - \frac{c}{r}\right)\right) = c - r$$

It can be interpreted as:

If the outflow rate is less than the growth rate of the bacteria,

i.e. $c < r$ then, the equilibrium point $P^* = 0$ is unstable where as $P^* = K \left(1 - \frac{c}{r}\right)$ is stable.
If $s > r$, then $P^* = 0$ is stable where as the equilibrium point $P^* = K \left(1 - \frac{c}{r}\right)$ is unstable because it is not biologically relevant since it is negative.

The exact solution of the chemostat equation (3.1) is

$$P = \frac{K \left(1 - \frac{c}{r}\right) P_0}{P_0 + \left(K \left(1 - \frac{c}{r}\right) - P_0\right) e^{-(r-c)t}}$$

If $c < r$ and $P_0 < K \left(1 - \frac{c}{r}\right)$ but well near to $K \left(1 - \frac{c}{r}\right)$, it will remain smaller than $K \left(1 - \frac{c}{r}\right)$ and will converge to it as $t \rightarrow \infty$.

We have discussed for the following three possible cases for analysis of the model equation (3.1) in fuzzy environment:

- I. Initial population is a fuzzy number.
- II. Growth rate in the chamber and outflow rate from the chamber are fuzzy numbers.
- III. Initial population as well as growth rate and outflow are fuzzy numbers.

We start with the following new theorems:

Theorem 3.1. *All the equilibrium points of the system of equations*

$$(3.2) \quad \begin{cases} \frac{dx}{dt} = ax \left(1 - \frac{y}{K}\right) - by \\ \frac{dy}{dt} = cy \left(1 - \frac{x}{K}\right) - dx \end{cases}$$

where a, b, c, d and K are all positive constants, are unstable.

Proof: If (x^*, y^*) is an equilibrium point of the system (3.2), then $\frac{dx^*}{dt} = 0$ and $\frac{dy^*}{dt} = 0$. On simplification we get $(0, 0)$ and $E_1 = \left(K \frac{(ac-bd)}{a(c+d)}, K \frac{(ac-bd)}{a(c+d)} \right)$ as the equilibrium points of the system (3.2). So that the Jacobian matrix is

$$J_1 = \begin{bmatrix} a \left(1 - \frac{y}{K}\right) & -\frac{ax}{K} - b \\ -\frac{cy}{K} - d & c \left(1 - \frac{x}{K}\right) \end{bmatrix}.$$

Now, the Jacobian matrix at point $(0, 0)$ is

$$J_1(0, 0) = \begin{bmatrix} a & -b \\ -d & c \end{bmatrix}.$$

The characteristic equation of $J_1(0, 0)$ is

$$\lambda^2 - p\lambda + q = 0$$

where

$$p = \text{Tr}(J_1(0, 0)) = a + c$$

and

$$q = \text{Det}(J_1(0, 0)) = ac - bd.$$

If $p < 0$ and $q > 0$, then by Routh-Hurwitz criteria, we find that the system is stable but here $p = a + c > 0$, so the equilibrium point $(0, 0)$ is unstable.

Now, at equilibrium point E_1 , the Jacobian matrix is

$$J_1(E_1) = \begin{bmatrix} \frac{ab(c+d)}{c(a+b)} & -\frac{c(a+b)}{c+d} \\ -\frac{a(c+d)}{c+d} & \frac{cd(a+b)}{a(c+d)} \end{bmatrix}.$$

Again, the characteristic equation of $J_1(E_1)$ is given by

$$\lambda^2 - p_1\lambda + q_1 = 0$$

where $p_1 = \text{Tr}(J_1(E_1))$ and $q_1 = \text{Det}(J_1(E_1))$.

Here, $p_1 = \text{Tr}(J_1(E_1)) = \frac{ab(c+d)}{c(a+b)} + \frac{cd(a+b)}{a(c+d)} > 0$ so, by Routh-Hurwitz criteria, we see that the equilibrium point E_1 is unstable.

If we interchange the variables x and y in the left hand side of the system of differential equations in the Theorem 3.1, we have the following result:

Theorem 3.2. *Among all the equilibrium points of the system of equations*

$$(3.3) \quad \begin{cases} \frac{dy}{dt} = ax \left(1 - \frac{y}{K}\right) - by \\ \frac{dx}{dt} = cy \left(1 - \frac{x}{K}\right) - dx \end{cases}$$

where a, b, c, d and K are all positive constants,

- (1) The trivial equilibrium point $(0, 0)$ is stable if $ac < bd$, and
- (2) The co-existential equilibrium point is stable if $ac > bd$.

The proof is as similar as in Theorem 3.1.

We consider the model equation when the initial population is a fuzzy number and the solution is as follows:

Let $\tilde{P}(0) = \tilde{P}_0$ is a triangular fuzzy number and let us consider $(\tilde{P}(t))_\alpha = [\underline{P}(t), \overline{P}(t)]$ is the α -cut of the solution. Then, the following two sub cases arises:

Case I: $\tilde{P}(t)$ is (i) – gH differentiable.

Then, the model equation (3.1) becomes

$$(3.4) \quad \begin{cases} \frac{d\underline{P}}{dt} = r\underline{P} \left(1 - \frac{\overline{P}}{K}\right) - c\overline{P} \\ \frac{d\overline{P}}{dt} = r\overline{P} \left(1 - \frac{\underline{P}}{K}\right) - c\underline{P} \end{cases}$$

So that, by Theorem 3.1, we can claim that all the equilibrium points of the system (3.4) are unstable.

Case II: $\tilde{P}(t)$ is (ii)- gH differentiable.

Then, model equation (3.1) becomes

$$(3.5) \quad \begin{cases} \frac{d\overline{P}}{dt} = r\underline{P} \left(1 - \frac{\overline{P}}{K}\right) - c\overline{P} \\ \frac{d\underline{P}}{dt} = r\overline{P} \left(1 - \frac{\underline{P}}{K}\right) - c\underline{P} \end{cases}$$

The system of equations (3.5) have two equilibrium points $(0, 0)$ and $(K(1 - \frac{c}{r}), K(1 - \frac{c}{r}))$. Using the Theorem 3.2, we can claim that the trivial equilibrium point $(0, 0)$ of the system is stable if $c > r$ and non-trivial equilibrium point $(K(1 - \frac{c}{r}), K(1 - \frac{c}{r}))$ is stable if $r > c$.

The four graphs in Figure 1 represent the solutions of the chemostat equations (3.1) when \tilde{P} is (ii) – gH differentiable for $r = 0.1, K = 100$, the fuzzy initial value $\tilde{P}(0) = [20, 25, 30]$ and $c = 0.05$ for $r > c$ and $c = 0.15$ for $r < c$. The crisp solution lies between the fuzzy solutions and fuzzy solutions are closure to crisp solution when α is increased from towards one and collide when $\alpha = 1$. Therefore, the fuzziness is well verified and solution so obtained is the strong fuzzy solution.

Now, we consider the growth rate and outflow rates as fuzzy numbers which are given by following cut sets $\tilde{r} = [\underline{r}, \overline{r}]$ and $\tilde{c} = [\underline{c}, \overline{c}]$. Then, we have the following two cases:

Case I: $\tilde{P}(t)$ is (i)- gH differentiable.

Then, the model equation (3.1) implies the following

$$(3.6) \quad \begin{cases} \frac{d\underline{P}}{dt} = \underline{r} \cdot \underline{P} \left(1 - \frac{\overline{P}}{K}\right) - \overline{c} \cdot \overline{P} \\ \frac{d\overline{P}}{dt} = \overline{r} \cdot \overline{P} \left(1 - \frac{\underline{P}}{K}\right) - \underline{c} \cdot \underline{P} \end{cases}$$

So, by Theorem 3.1 the system of differential equations is unstable for every equilibrium points.

Case II: $\tilde{P}(t)$ is (ii)- gH differentiable.

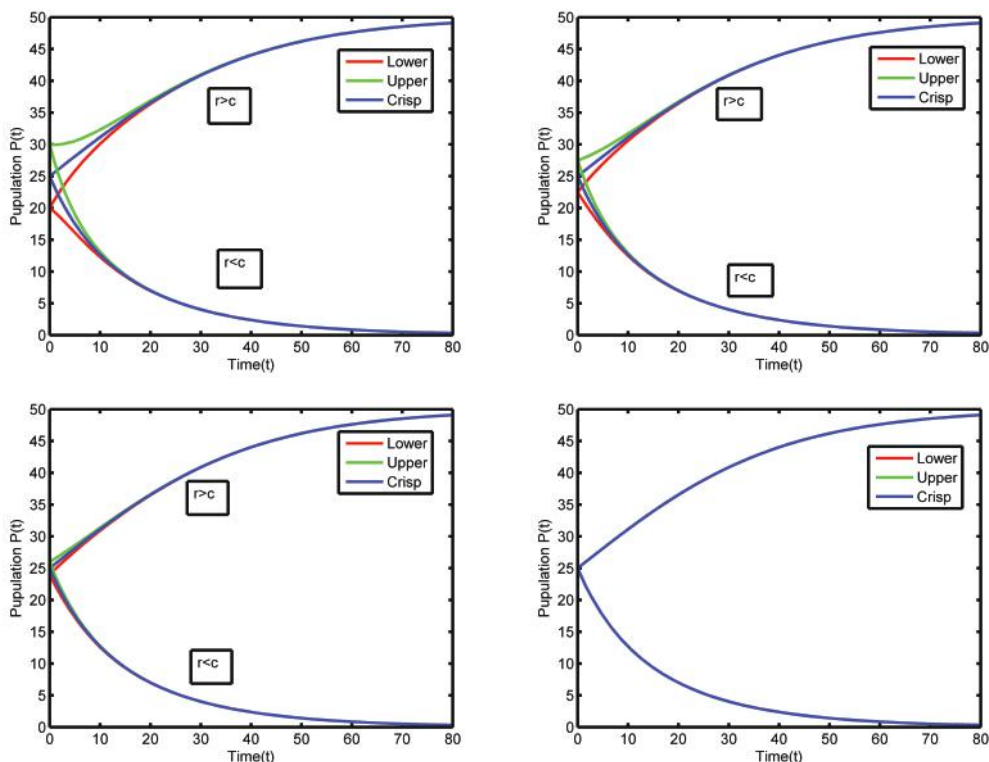


FIGURE 1. Graphs represented by chemostat equations (3.5) for different cut sets when α to be 0, 0.5, 0.8 and 1 respectively.

Then, the model equation (3.1) is transformed into

$$(3.7) \quad \begin{cases} \frac{d\bar{P}}{dt} = r.\underline{P} \left(1 - \frac{\bar{P}}{K} \right) - \bar{c}.\bar{P} \\ \frac{d\underline{P}}{dt} = \bar{r}.\bar{P} \left(1 - \frac{\underline{P}}{K} \right) - \underline{c}.\underline{P} \end{cases}$$

So, the system (3.7) has the trivial equilibrium point $(0, 0)$ which is stable if $\underline{c}\bar{c} > \underline{r}\bar{r}$ and non-trivial equilibrium point $\left(K \frac{(\underline{r}\bar{r} - \underline{c}\bar{c})}{\underline{r}(\bar{r} + \underline{c})}, K \frac{(\underline{r}\bar{r} - \underline{c}\bar{c})}{\bar{r}(\underline{r} + \bar{c})} \right)$ which is stable if $\underline{c}\bar{c} < \underline{r}\bar{r}$.

The four graphs in Figure 2 represent fuzzy solutions of the chemostat equations (3.1) when \tilde{P} is $(ii) - gH$ differentiable for $P(0) = 35, K = 100, \tilde{r} = [0.096, .1, .104]$ and $\tilde{c} = [0.046, 0.05, 0.054]$ for $\underline{r}\bar{r} > \underline{c}\bar{c}$ and $\tilde{c} = [0.11, 0.15, 0.19]$ for $\underline{r}\bar{r} < \underline{c}\bar{c}$. The crisp solution lies between the fuzzy solutions and fuzzy solutions are closure to crisp solution when α is increased from towards one and collide when $\alpha = 1$. Therefore, the fuzziness is well verified and solution so obtained is the strong fuzzy solution.

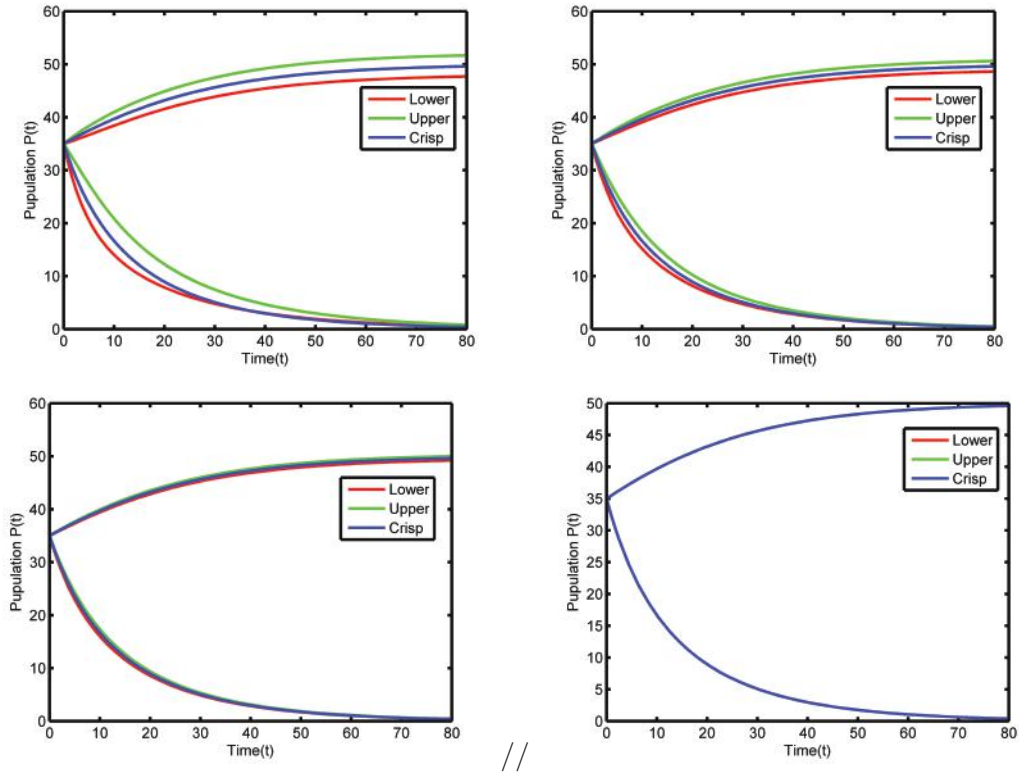


FIGURE 2. Graphs represented by chemostat equations (3.7) for different cut sets when α is 0, 0.5, 0.8 and 1 respectively.

Now, consider the case when the initial population as well as growth rate and immigration rates are fuzzy numbers. The stability condition in this section is exactly same as previous in which intrinsic growth rate and immigration constant are fuzzy numbers. Therefore, only numerical verification is presented here. The four graphs in Figure 3 represent the fuzzy solutions of the chemostat equations (3.1) when \tilde{P} is *(ii)* - gH differentiable for $P(0) = [20, 25, 30]$, $K = 100$, $\tilde{r} = [.096, .1, .104]$ and $\tilde{c} = [0.046, 0.05, 0.054]$ for $\underline{r}\bar{r} > \underline{c}\bar{c}$ and $\tilde{c} = [0.11, 0.15, 0.19]$ for $\underline{r}\bar{r} < \underline{c}\bar{c}$. The crisp solution lies between the fuzzy solutions and fuzzy solutions are closure to crisp solution when α is increased from towards one and collide when $\alpha = 1$. Therefore, the fuzziness is well verified and solution so obtained is the strong fuzzy solution.

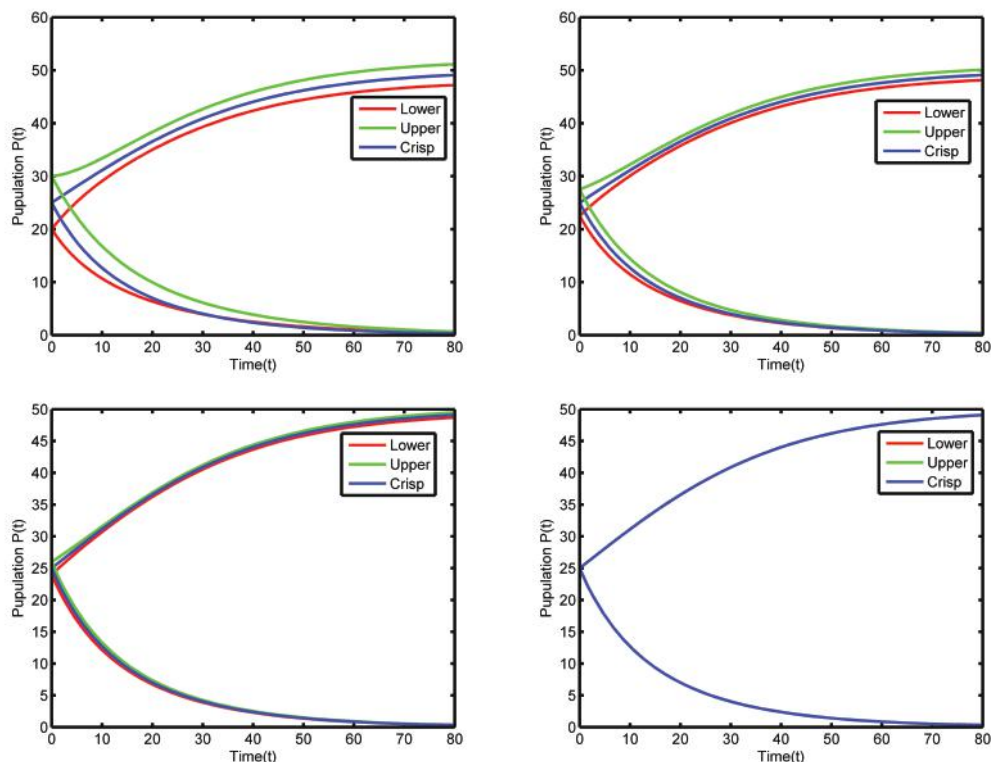


FIGURE 3. Graphs represented by chemostat equations (3.5) with fuzzy initial value for different cut sets when α to be 0, 0.5, 0.8 and 1 respectively.

4. CONCLUSION

Our paper is the extension of the results of Paul et.al[14] in terms of harvesting function and transformation of the immigration function by V. Kumar and S. Lal [15] in fuzzy environment. Our results explain the fuzzy phenomena of the microbial population growth model in chemostat vessel. Also, some new results show the stability of chemostat model for gH-differentiability and have been successfully applied. The results so obtained are justifiable analytically and verified graphically.

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