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An Uncertainty Principle like Hardy's Theorem for Nilpotent Lie Group G_n

CHET RAJ BHATTA
Central Department of Mathematics
Tribhuvan University, Kirtipur
Email: crbhatta@yahoo.com

Abstract

Let f: $\Re \to \mathbb{C}$ be measurable and for all x, y $\in \Re$ and if

(i) $|f(x)| \leq C \exp(-a\pi x^2)$

(ii) $|\hat{f}(y)| \leq C \exp(-b\pi y^2)$

where C, a, b > 0. If ab > 1 then f = 0 a.e. If ab = 1 then f(x) = C exp (−$a\pi x^2$). Iϕ $\alpha\beta < 1$ then there exist infinitely many linearly independent function satisfying (i) and (ii).

In this paper we extend an uncertainty principle due to Cowling and Price to threadlike nilpotent Lie groups G_n.

Keywords and Phrases: Uncertainty principle, nilpotent Lie group, Fourier transform, Hilbert Schmidt norm.

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Introduction

A classical theorem of Hardy [6] on Fourier transform pairs says that a non-zero function f on the real line $\Re$ and its Fourier transform $\hat{f}$ can not both be very rapidly decreasing. More precisely, let the Fourier transform be defined by,

$$\hat{f}(y) = \int_{\Re} f(x) \exp(-2\pi i x y)\, dx, \; y \in \Re$$

The following is a generalization of this theorem due to Cowling and Price [3].

Theorem (Cowling and Price): Let f:$\Re \to \mathbb{C}$ be measurable and

(i) $\|e_a f\|_{L^p(\Re)} < \infty$

(ii) $\|e_b \hat{f}\|_{L^q(\Re)} < \infty$

Where a, b > 0, $e_k(x) = \exp(k\pi x^2)$ and $1 \le \min(p, q) < \infty$. If $ab \ge 1$, then f = 0 a.e. If ab < 1, then there exist infinitely many linearly independent functions satisfying (i) and (ii)

An analogue of the Cowling-Price theorem has been proved in [3] for Eucledian space the Heisenberg group H_n and the Eucledian motion group of the plane.

Main Results

Theorem: let $f:G_n \to \mathbb{C}$ be a measurable function such that $|f(x)| \le C \exp(-a\pi \|x\|^2)$ for some c, a > 0 and all $x \in G_n$ then the function $h(\xi) = \|\pi_\xi(f)\|_{HS}$ is bounded.

Proof:

$$|\xi_1|\,\|\pi_\xi(f)\|^2_{HS} = \int_{\Re^2} |\mathcal{F}_1 \ldots (n-1)\, f(\xi_1, t, q_3(\xi, t), q_{n-1}(\xi, t), s)|^2\, dt\, ds$$

$$\le \int_{\Re^{n-1}} |f(x_1, \ldots x_{n-1}, s)|\, |e^{2\pi i \xi_1 x_1}|\, |e^{2\pi i t x}2|\, dx_1 dx_2 \ldots dx_{n-1}$$

$$\le C \int_{\Re^{n-1}} \exp(-a\pi(x_1^2 + \ldots + x_n^2 + s^2)\,|\exp(2\pi i\, \xi_1 x_1)|\, |\exp(2\pi i x_2)|\, dx_1 \ldots dx_{n-1}$$

$$\le \text{constant} \exp(-a\pi s^2) \int_{\Re^2} |\exp(-a\pi(x_1^2 - \frac{2}{a} i\, \xi_1 x_1)|\, |\exp\left(-a\pi\left(x_2^2 - \frac{2}{a} itx_2\right)\right| dx_1\, dx_2$$

$$\int_{-\infty}^{\infty} |\exp\left(-a\pi\left(x^2 - \frac{2}{a} i\, \xi_1 x\right)\right| dx = \int_{-\infty}^{\infty} |\exp(-a\pi(x+iy)^2 - \frac{2}{a} i\, \xi_1(x+iy)|\, dx$$

By the change of contour $x \to x + iy$

$$= \int_{-\infty}^{\infty} \exp \mathrm{Re}\,(-a\pi((x+iy)^2 - \frac{2}{a} i\, \xi_1(x+iy))\, dx$$

$$= e^{-2\pi \xi_1 y} \int_{-\infty}^{\infty} \exp(-a\pi(x^2 - y^2))\, dx$$

$$= e^{-2\pi}(\xi y - \frac{a}{2} y^2) \int_{-\infty}^{\infty} \exp(-a\pi x^2) dx$$

$$\le \text{const.}\; e^{-2\pi}(\xi, y - \frac{a}{2} y^2)$$

Taking infimum over y, we have

$$\int_{-\infty}^{\infty} |\exp(-a\pi(x^2 - \frac{2}{a} i\xi_1 x)|\, dx \le \text{const}\ e^{-2\pi} \sup\left(\xi_1\, y - \frac{a}{2}\, y^2\right)$$

$$= \text{const}\ e^{-2\pi\xi_1^2/2a}$$

$$= \text{const}\ e^{-\pi\xi_1^2/a}$$

$$\therefore\quad \|\pi_\xi(f)\|_{HS} \le \text{const}\ (|\xi_1|^{-1} \exp(-\pi\xi_1^{\,2}/a))^{1/2}$$

Thus h is bounded.

The sharpness of the constant

Let G_4 be the low dimensional nilpotent Lie groups for $ab < 1$, let $\alpha \in (a, 1/b)$, but it works for G_n using lemms 2.1, 2.2 in [7]

For $x \in G_4$, let $f(x) = \exp(-\alpha\pi\, \|x\|^2)$

We know that,

$$\|\pi\xi_1, \xi_2(f)\|^2_{HS} = |\xi_1)^{-1} \int_{\Re} |\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v)|^2\, du\, dv$$

For $u, v \in \Re$,

$$\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v) = \int_{\Re^3} e^{-\propto\pi(x_1^2 + x_2^2 + x_3^2 + v^2)}\, e^{-2\pi i x_1 \xi_1}\, e^{-2\pi i x_2 u} e^{-2\pi i x_3 \{\xi_3 + (1/2)(u^2/\xi_1)\}}\, dx_1 dx_2 dx_3$$

$$= e^{-\propto\pi v^2} \mathcal{F}(e^{-\propto\pi x_1^2})(\xi_1) \mathcal{F}(e^{-\propto\pi x_2^2})\,(u)\, \mathcal{F}(e^{-\propto\pi x_3^2})\ \{\xi_3 + (1/2)\,(u^2/\xi_1)\} \quad \ldots (A)$$

$$= e^{-\propto\pi v^2} \mathcal{F}(e^{-\propto\pi x_1^2})(\xi_1) \mathcal{F}(e^{-\propto\pi x_2^2})\,(u)\, \mathcal{F}(e^{-\propto\pi x_3^2 - 2\pi i \xi_3 x_3})\ \{(1/2)\,(u^2/\xi_1)\}$$

where $\mathcal{F}(e^{-\propto\pi x_1^2})\,(\xi_1)$ is the Fourier transform of the function $g(x) = e^{-\propto\pi x^2}$ at ξ_1

Now, $\mathcal{F}(e^{-\propto\pi x_1^2})\,(\xi_1) = \text{const}\ e^{-\propto\pi x_1^{2/\propto}}$

And $\mathcal{F}(e^{-\propto\pi x_2^2})(u) = \text{const}^{-\pi u_2/\propto}$

So, $|\mathcal{F}_{123} f(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_3, v)|$

$$= \text{const.}\ e^{(-\pi\xi_1^{2})/\propto}\, e^{(-\pi u_2)/\propto} e^{-\propto\pi v^2} |\mathcal{F}(e^{-2\pi i \xi_3 x_3 - \propto\pi x_3^2}) \left(\frac{1}{2}\frac{u^2}{\xi_1}\right)|$$

$$\le \text{const.}\ e^{(-\pi\xi_1^{2})/\propto} e^{(-\pi u^2)/\propto}\ e^{-\pi u v^2} \int_{\Re} |\ e^{-2i\pi i x_3 x_3 - \propto\pi x_3^2}|\, dx_3$$

$$= \text{const. } e^{(-\pi\xi_1{}^2)/\alpha} e^{(-\pi u^2)/\alpha} e^{-\pi uv^2} e^{(-\pi\xi_3{}^2)/\alpha} \int_{\Re} |e^{-\alpha\pi\{x_3 + (i\xi_3/\alpha)^2\}^2}| dx_3$$

$$= \text{const. } e^{(-\pi\xi_1{}^2)/\alpha} e^{(-\pi u^2)/\alpha} e^{-\pi uv^2} e^{(-\pi\xi_3{}^2)/\alpha}$$

$$\therefore \quad \|\pi\xi_1\xi_3(f)\|^2_{HS} \leq \text{const } |\xi_1|^{-1} e^{-2\pi(\xi_1{}^2 + \xi_3{}^2)/\alpha} \int_{\Re} e^{-2\pi u^2/\alpha} e^{-2\pi\alpha v^2} \, du \, dv$$

$$= \text{const. } |\xi_1|^2 e^{-2\pi(\xi_1{}^2 + \xi_3{}^2)/\alpha} \qquad \ldots (B)$$

$$\int_{\Re^2} e^{2\pi b(\xi_1{}^2 + \xi_2{}^2)} |\xi_1| \, \|\pi\xi_1, \xi_3 (f)\|^2_{HS} \, d\xi_1 \, d\xi_3$$

$$\leq \text{const } \int_{\Re^2} e^{2\pi (b - 1/\alpha)(\xi_1{}^2 + \xi_3{}^2)} \, d\xi_1 \, d\xi_3$$

$$< \infty \text{ since } b - \frac{1}{\alpha} < 0$$

Thus there is a non zero function satisfying the condition (i) and (ii) of theorem 2.4 [7] for $ab < 1$.

Since $|f(x)| = \exp(-\alpha\pi x^2) < \exp(-a\pi x^2)$ for $q = 2$.

In case the condition (ii) is replaced by

$$\int_{\Re^2} e^{2\pi b\xi_1{}^2} |\xi_1| \, \|\pi_{\xi_1, \xi_3} (f)\|^2_{HS} \, d\xi_1 \, d\xi_3 < \infty$$

which is really the condition used in the proof of $q = 2$, the things are more simpler.

Write the equation (A) as

$$\mathcal{F}_{123} f\left(\xi_1, u, \frac{1}{2}\frac{u^2}{\xi_1} + \xi_{3,v}\right)$$

$$= \text{const } e^{-\alpha\pi v^2} e^{(-\pi\xi_1{}^2)/\alpha} e^{(-\pi u_1{}^2)/\alpha} e^{-(\pi/\alpha)\{\xi_3 + (1/2)(u^2/\xi_1)\}^2}$$

$$\therefore \|\pi_{\xi_1, \xi_2} (f)\|^2_{HS} = \text{const. } |\xi_1|^{-1} e^{(-2\pi\xi_1{}^2)/\alpha} \int_{\Re^2} e^{-2\alpha\pi v^2} e^{-2\pi/\alpha[u^2 + \{\xi_3 + (1/2)(u^2/\xi_1)\}^2]} \, du \, dv$$

$$= \text{const. } |\xi_1|^{-1} e^{(-2\pi\xi_1{}^2)/\alpha} \int_{\Re} e^{-2\pi/\alpha[u^2 + (\xi_3 + 1/2\, u^2/\xi_1)^2]} \, du$$

$$\int_{\Re^2} e^{-2\pi b\xi_1{}^2} |\xi_1| \, \|\pi_{\xi_1, \xi_3} (f)\|^2_{HS} \, d\xi_1 \, d\xi_3$$

$$= \text{const. } \int_{\Re^3} e^{2\pi(b - 1/\alpha)\xi_1{}^2} e^{(-2\pi u^2)/\alpha} e^{-2\pi/\alpha\{\xi_3 + (1/2)(u^2/\xi_1)\}^2} \, du \, d\xi_1 \, d\xi_3$$

Applying $\xi_3 \to \xi_3 - \dfrac{1}{2}\dfrac{u^2}{\xi_1}$

$$= \text{const.} \int_{\Re^3} e^{2\pi\{b-1/\alpha\}\,\xi_1^2}\, e^{(-2\pi u^2)/\alpha}\, e^{-2\pi/\alpha\,\xi_3^2}\, du\, d\xi_1\, d\xi_3$$

$$< \infty$$

For $q \geq 2$ or $1 \leq q < 2$ and $ab < 1$.

$$\int_{\Re^2} e^{qb\pi\{\xi_1^2+\xi_3^2\}}\, |\xi_1|\, \|\pi_{\xi_1,\xi_3}(f)\|_{HS}^q\, d\xi_1\, d\xi_3 \qquad \ldots(1)$$

$$\int_{\Re^2} e^{qb\pi\{\xi_1^2+\xi_3^2\}}\, |\xi_1|^{\{1-(q/2)\}}\, e^{-(ap/\infty)(x_1^2+x_3^2)}\, d\xi_1\, d\xi_3$$

$$= \int_{\Re^2} e^{\pi q(b-1/\alpha)(\xi_1^2+\xi_3^2)}\, |\xi_1|^{1-q/2}\, dx_1\, d\xi_3 < \infty \text{ for } 1 \leq q < 2$$

and for $q \geq 2$ considering the case $|\xi_1| > 1$ we obtain integral in (i) is finite.

Theorem: Let a and b be positive real numbers and $1 \leq \min(p, q) < \infty$. Suppose that

$f \in L^1(Gn) \cap L^2(Gn)$ satisfies the following conditions:

(i) $\int_{Gn} e^{pa\pi\|\xi\|^2}\, |f(x)|^p\, dx < \infty$

(ii) $\int_{\Re^{n-2}} |\xi_1|\, e^{b\pi q\|\xi\|^2}\, \|\pi_\xi(f)\|_{HS}^q\, d\xi < \infty$

If $1 \leq q < 2$ and $ab \geq \frac{2}{q}$ then $f = 0$ a.e.

Proof: Let p be such that $\frac{1}{p} + \frac{1}{q} = 1$. Clearly $p \geq 2$ and $q < p$ by Lemma 2.2 [7] we have

$$\|e_{b\{1+q/p\}}\, \hat{g}\|_1 = \int_{\Re} e_{b\{1+(q/p)\}}(\xi_1)\, |\hat{g}(\xi_1)|\, d\xi_1$$

$$= \int_{\Re^{n-2}} |\xi_1|\, e_{b\{1+(q/p)\}}(\xi_1)\, \|\pi_\xi(f)\|^2\, d\xi_1\, d\xi_3 \ldots d\xi_{n-1}$$

Define the function u and v in $\Re^{n-2}$ by

$u(\xi) = e_b(\xi_1)\, |\xi_1|^{1/q}\, \|\pi_\xi(f)\|_{HS}$ and

$v(\xi) = e_{bq/p}(\xi_1)\, |\xi_1|^{1/p}\, \|\pi_\xi(f)\|_{HS}$

Then,

$$\int_{\Re^{n-2}} |u(\xi)|^q\, d\xi = \int_{\Re^{n-2}} e_{bq}(\xi_1)\, |\xi_1|\, \|\pi_\xi(f)\|_{HS}^q\, d\xi < \infty$$

and

$$\int_{\Re^{n-2}} |v(\xi)|^p\, d\xi = \int_{\Re^{n-2}} e_{bq}(\xi_1)\, |\xi_1|\, \|\pi_\xi(f)\|_{HS}^p\, d\xi$$

$$= \int_{\Re^{n-2}} e_{bq}(\xi_1)\,|\xi_1|\,\|\pi_\xi(f)\|_{HS}^{q}\,\|\pi_\xi(f)\|_{HS}^{p-q}\,d\xi \quad (p > q)$$

$$\le K \int_{\Re^{n-2}} e_{bq}(\xi_1)\,|\xi_1|\,\|\pi_\xi(f)\|_{HS}^{q} > \infty$$

So using Holders inequality we have

$$\|e_{b(1+(q,p))}\hat{g}\|_1 \le \|u\|_q \|v\|_p < \infty$$

Since $\frac{a}{2} b\left(1 + \frac{q}{p}\right) \ge \frac{1}{2} \cdot \frac{2}{q}\left(1 + \frac{q}{p}\right) = \frac{1}{2} \cdot \frac{2}{q} \cdot q = 1$

The Cowling price theorem shows that $g = 0$ and hence $f = 0$ a.e.

REFERENCES

[1] Bagchi, S.C. and Ray S.K. Uncertainty Principle like Hardy's theorem on some Lie groups, J. Austral Math. Soc. (Series A) 65 (1998), 289–302.

[2] Bhatta, C.R., Cowling Price Theorem for Low Dimensional Nilpotent Lie groups G_4 (II), Journal of Institute of Science and technology, Vol. 16, 2009–10, 130–134.

[3] Corwin, L. and Greenleaf, F.P., representations of nilpotent Liegroups and their applications, Part I, Basic theory and examples (Cambridge University Press, 1990).

[4] Dym H. and Mckean H.P., Fourier series and Integrals (Academic Press, New York, 1972).

[5] Folland, G.B., A Course in abstract harmonic analysis (CRC Press, Boca Ratio, 1995).

[6] Hardy, G.H., A theorem concerning Fourier transforms, J. London Math. Soc. 9(1934), 187-192.

[7] Kumar, Ajay & Bhatta, C.R., An Unvertainty Principle like Hardy's theorem for nilpotent Lie groups, J. Aust. Math. Soc. 77(2004).

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