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Any Empty Spacetime Has Not Constant Timelike Vectors

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Abstract: Using a Synge's invariant we show that any vacuum geometry accepts constant timelike vectors.

1. Introduction.

Synge [1] deduced an invariant with the property that, when it is zero at a particular event of the spacetime then the curvature tensor is also zero at these event. This result of Synge is surprising because it means that one zero valued scalar implies nullification of the twenty independent components of the Riemann tensor. Thus we have the :

Theorem, " Let τ_r be any unitary timelike vector and

$$H = \frac{1}{20} H_0 + \frac{23}{60} H_2 + \frac{30}{10} H_4 . \tag{1}$$

with

$$H_0 = \frac{3}{2} R_{abcd} R^{abcd} + 2 R_{ab} R^{ab} + \frac{1}{2} R^2 ,$$

$$H_2 = 2 \left[R_{pqra} \left(R^{pqrb} + R^{rqpb} \right) - R^{pq} R^b_{paq} + R^p_a R^b_p + \frac{R}{2} R^b_a \right] \tau^a \tau_b . \tag{2}$$

$$H_4 = \left[R^{paqb} R_{pcqd} + \frac{1}{2} R^{ab} R_{cd} \right] \tau_a \tau_b \tau^c \tau^d ,$$

then for each event in R_4 :

(3) $$H = 0 \Rightarrow R_{ijkm} = 0,"$$

We remember than $R_{ab} = R^{c}_{abc}$ is the Ricci tensor and $R = R^{a}_{a}$ is the scalar curvature. It is clear that $H = 0$ in every point of R_4 implies a flat spacetime,

This Synge's theorem can be very useful in general relativity : we give here one application of it showing that a vacuum metric not admits constant timelike vectors.

2. Empty Spacetime.

The ref. [2] motivates us to recognize the importance of the existence of constant vectors :

(4) $$\tau^{r}{}_{,c} = 0.$$

Thus the non-commutative property of covariant derivative leads to :

(5) $$R^{abcd}\ \tau_d = 0$$

In this Section we consider the case $R_{ab} = 0$, and with the use of H and (5) we show that there are no constant timelike vectors in empty spacetime. This can be applied to several vaccum geometries such as those of Taub, Schwarzschild, C. Siklos, Kerr, etc.

In fact we suppose there exists a unitary constant timelike vector τ^{r} (note there is no loss of generality because the norm of a constant vector is also a constant), then from (1), (2) and (5) we obtain :

(6) $$H_0 = \frac{3}{40} R_{abcd}\ R^{abcd}$$

But in [2] it was proved (using a result of Horndeski [3]) that the Lanczos scalar [4] $R_{abcd}\ R^{abcd}$ for an empty spacetime is zero in presence of a non-null constant vector. Therefore $H = 0$ and thus (3) affirms that our R_4 is flat, that is, only the Minkowski spacetime admits constant timelike vectors q.e.d.

The proof here presented is simpler than other ones, for example, in [5] special coordinate systems are required.

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