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Applications of the Class *SDCP* with Plane Harmonic Mappings

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Abstract : In [2] we introduced and fully characterized the class *SDCP* (Strongly Direction Convexity Preserving). In this paper we shall discuss its applications to harmonic mappings in plane.

1. Introduction to the class *SDCP*

Let A denote the set of analytic functions in $\mathbb{D}$. Let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty} b_n z^n$$

be two members of A. Then the Hadamard product or convolution between f and g, denoted by $f * g$, is defined by

$$(f * g)(z)=\sum_{n=0}^{\infty} a_n b_n z^n .$$

A domain $D \subset \mathbb{C}$ is said to be *convex in the direction* $e^{i\varphi}$, $\varphi \in \mathbb{R}$, if and only if for every $a \in \mathbb{C}$ the set

$$D \cap \{ a+te^{i\varphi} : t \in \mathbb{R} \}$$

is either connected or empty. Accordingly we define the classes $K(\varphi) \subset A$, $\varphi \in \mathbb{R}$, of the functions *convex in the direction* $e^{i\varphi}$ as

$$K(\varphi) := \{ f \in A : f \text{ univalent and } f(\mathbb{D}) \text{ convex in the direction } e^{i\varphi} \}.$$

Finally, a function $g \in A$ is called *Direction-Convexity-Preserving* ($g \in DCP$) if and only if

$$g * f \in K(\varphi) \text{ for all } f \in K(\varphi) \text{ and all } \varphi \in \mathbb{R}$$

Functions in *DCP* have many intriguing convolution-type properties, for instance the preservation of convex harmonic functions in $\mathbb{D}$, and of Jordan curves in the plane with convex interior domain we refer to [13],[14] for more details. There one also finds a complete description of the members of *DCP*, namely

$$g \in DCP \Leftrightarrow g(z) + itzg'(z) \in k\left(\tfrac{\pi}{2}\right) \text{ for all } t \in \mathbb{R}.$$

Further it is known, that *DCP* functions are convex univalent. The class *DCP* is not rotation invariant. That is, $f \in DCP$ does not always imply that $e^{i\varphi} f$ is in *DCP*, $\forall \alpha \in \mathbb{R}$. However, $f \in DCP \Rightarrow Af + B \in DCP$ for all $A : \mathbb{R} \setminus \{0\}$ and $B \in \mathbb{C}$. Hence we can normalize the class *DCP* by the conditions $f(0) = 0$ and $|f'(0)| = 1$. Motivated by the non-existent rotation invariance property of the class *DCP*, we are interested in studying the subclass of those functions in *DCP* which are rotation invariant.

Definition 1.1. A function $f \in A$ is said to be in the class *SDCP* (Strongly Direction Convexity Preserving) if and only if $e^{i\alpha} f \in DCP$ for all $\alpha \in \mathbb{R}$

Every $f \in SDCP$ is univalent and hence fulfills $f'(0) \neq 0$. At the same time, it is clear that

$$\frac{f(z) - f(0)}{f'(0)} \in SDCP.$$

Hence we can normalize the class *SDCP* by the conditions $f(0) = 0$ and $f'(0) = 1$. From now on, by the class *SDCP*, we always mean the normalized class defined as follows :

$$SDCP := \{ f \in S : e^{i\alpha} f \in DCP, \forall \alpha \in \mathbb{R} \}.$$

We refer [2] for detail about the class *SDCP*.

2. Introduction to Harmonic Mappings in the Plane

A complex valued function $f(x,y)$ is harmonic in a domain D in the plane if it satisfies the Laplace's equation $\Delta f = 0$, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The complex notation $z = x + iy$ for points in the plane leads to the differential operators

(1) $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

(2) $$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

A simple calculation gives

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} = 4\frac{\partial^2 f}{\partial \bar{z} \partial z},$$

so the Laplace's equation can be written

(3) $$f_{\bar{z}z} = \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0.$$

For a complex-valued function with continuous first order partial derivatives, the equation $f_{\bar{z}} = 0$ is equivalent to Cauchy-Riemann equations. Thus a function f is analytic if and only if $f_{\bar{z}} = 0$. From (3) it follows that every analytic function is harmonic, and that the z-derivative of every harmonic function is analytic.

The word harmonic mapping will be reserved for univalent (one-to-one) function. A complex–valued harmonic function f is a harmonic mapping of a domain $D \subset \mathbb{C}$ if it maps D unvalently on to some planer domain $\Omega = f(D)$. Although harmonic mappings are natural generalization of conformal mappings, they were studied originally by differential geometers because of their natural role in parametrizing minimal surfaces. Only in the mid 1980s did harmonic mappings begin to attract wide-spread interest among complex analysists. Particularly after the landmark paper [4] by J. Clunie and T. Sheil-Small which pointed out that many of the classical result for conformal mappings have clear analogues for harmonic mapping. Since that time the study of harmonic mapping developed rapidly, although a number of basic problems remain unsolved. In this paper we will only discuss applications of the class *DCP* with harmonic mappings in a plane.

If we write f as a sum of two real valued function $u(x,y)$ and $v(x,y)$ say, $f = u + i\,v$, the Jacobian of f is

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

In the case of an analytic function we have the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

so that $J_f = |f_x|^2 = |f_y|^2 = |f'|^2$. In the more general case of harmonic function, a simple calculation, with the help of (1) and (2), we can express the Jacobian J_f of f as

(4) $$J_f = |f_z|^2 = |f_{\bar{z}}|^2.$$

A sufficient condition for a differentiable function to be locally univalent (one-to-one) near a point $z_0 = x_0 + iy_0$ is that $J_f(z_0) \neq 0$. For analytic function it is also a necessary condition for locally univalence at z_0 as $J_f(z_0) = |f'(z_0)|^2 \neq 0$. Lewy [10] has shown that this property generalizes to harmonic function.

Lewy's Theorem : If f is a complex valued harmonic function which is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian J_f is different from zero for all $z \in \mathbb{C}$.

Thus a harmonic mapping is a domain D is either sense preserving with $J_f > 0$ or sense reversing with $J_f < 0$. More precisely a harmonic mapping is sense preserving if $|f_z(z)| > |f_{\bar{z}}(z)|$ for all $z \in D$ and in sense reversing if $|f_{\bar{z}}(z)| > |f_z(z)|$ for all $z \in D$. In particular if a harmonic mapping is sense preserving, then $f_z(z) \neq 0$ for all $z \in D$. A mapping is sense reserving if and only if its complex conjugate is sense preserving. Conformal mapping are sense-preserving.

If f is a complex valued harmonic function on a simply connected domain $D \subseteq \mathbb{C}$, then it has the representation.

(5) $$f = \overline{g} + h,$$

where g and h are analytic functions unique up to an additive constant in D, to justify this simple fact, first recall that since f is harmonic so $f_{z\bar{z}} = 0$. From this we see that f_z is analytic. Then let $h' = f_z$ and choose an antiderivative h. Let $g = \overline{g} - \overline{h}$ and observe that $g_{\bar{z}} = \overline{f_z} - \overline{h_z} = 0$, so that g is analytic. If 0 is in the domain D, we shall choose h so that $h(0) = f(0)$ and refer to $f = h + \overline{g}$ as the canonical representation of f. We call h the analytic and g the co-analytic part of f.

3. Application of the Class *SDCP* with Harmonic Mapping in the Plane

Let S_H denote the class of functions f of the form (5) that are univalent harmonic and sense preserving in the unit disk $\mathbb{D}$ and normalized by $f(0) = f_z(0) - 1 = 0$, and let S_H^0 denote the subclass of S_H for which $f_{\bar{z}}(0) = 0$. The class S_H obviously reduces to the familiar class S of normalized univalent functions in the unit disk D if the co-analytic part of f is zero.

In this section we shall look at various subclasses of S_H^0. We note that functions in S_H^0 have the form

(6) $$f = \overline{g} + h, \text{ where } h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; g(z) = \sum_{n=2}^{\infty} b_n z^n,$$

Let S_H^{*0} and C_H^0 be the subclasses of S_H^0 consisting of functions f that map $\mathbb{D}$ onto a starlike domain and convex domain, respectively, and let, as usual, C be the subclass of S consisting of those functions which map $\mathbb{D}$ onto a convex domain. In [7], the following theorem was proved.

Theorem 1. *If a function f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n\,(|a_n|+|b_n|) \le 1$, *then* $f \in S_H^{*0}$.

It has been proved in [3] that for $f \in S_H^{*0}$, the function $\int_0^z \left(\frac{f(\varsigma)}{\varsigma}\right) d\varsigma\ C_H^0$. Hence we have the following corollary :

Corollary 1. *If f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n^2\,(|a_n|+|b_n|) \le 1$, *then* $f \in S_H^0$..

Theorem 2. *If a function f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n\,(|a_n|+|b_n|) \le 1$, *then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in S_H^{*0}, \quad \forall\ \varphi \in C.$$

Proof : Let $\varphi(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$. Then the coefficients of φ satisfy the condition $|c_n| \le 1$. Therefore the coefficients of the function $\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$, whose analytic and co-analytic part have the form

$$\varphi * h = z + \sum_{n=2}^{\infty} c_n a_n z^n \ \text{ and } \ \varphi * g = \sum_{n=2}^{\infty} c_n b_n z^n.$$

respectively, satisfy the condition

$$\sum_{n=2}^{\infty} n(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n|c_n|\,(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le 1.$$

Hence by Theorem 5.4.1, the function $\varphi \,\tilde{*}\, f \in S_H^{*0}$

Corollary 2. *If f of the form* (6) *satisfies* $\sum_{n=2}^{\infty} n^2\,(|a_n|+|b_n|) \le 1$, *then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in C_H^{*0}, \quad \forall\ \varphi \in C.$$

Proof: Let the function φ be as in the previous theorem. Then the coefficients of the function

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$$

satisfy the condition

$$\sum_{n=2}^{\infty} n^2(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n^2|c_n|(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} n^2\,(|a_n| + |b_n|) \le 1.$$

Hence $\varphi \,\tilde{*}\, f \in S_H^0$, by Corollary 5.4.1.

Theorem 3. *If f of the form (6) satisfies $\sum_{n=2}^{\infty}(|a_n|+|b_n|)\le 1$, then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in S_H^{*0}, \quad \forall\, \varphi \in SDCP.$$

Proof : Let $\varphi(z) = z + \sum_{n=2}^{\infty} c_n z^n$. Since $\varphi \in SDCP$, the coefficients of φ satisfy the condition $|c_n| \le \frac{1}{n}$. Therefore the coefficients of the function $\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$, whose analytic and co-analytic part have the form

$$\varphi * h = z + \sum_{n=2}^{\infty} c_n a_n z^n, \quad \varphi * g = \sum_{n=2}^{\infty} c_n b_n z^n,$$

respectively, satisfy the condition

$$\sum_{n=2}^{\infty} n(|c_n a_n| + |c_n b_n|) = \sum_{n=2}^{\infty} n|c_n|\,(|a_n|+|b_n|) \le \sum_{n=2}^{\infty}(|a_n|+|b_n|) \le 1.$$

Hence by Theorem 5.4.1, the function $\varphi \,\tilde{*}\, f \in S_H^{*0}$.

We need the following result of J. Clunie and T.Shell-Small [4] for our next theorem

Lemma 1. Let $f = \overline{g} + h$ *be locally univalent and harmonic in* $\mathbb{D}$. *Then* f *is a univalent mapping of* $\mathbb{D}$ *onto a domain convex in the direction of the real axis if, and only if,* $h-g$ *is a conformal univalent mapping of* $\mathbb{D}$ *onto a domain convex in the direction of the real axis.*

Theorem 4. *Let* $C_H(\theta)$ *consist of functions* $f = \overline{g} + h$ *in* S_H *which are convex in the direction* $e^{i\theta}$, *Then*

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in C_H(\theta), \ \forall f \in C_H(\theta) \ \text{ and } \ \varphi \in DCP.$$

provided the function $\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$ *is locally univalent.*

Proof : Let $f = \overline{g} + h \in C_H(\theta)$. Then the function

$$e^{-i\theta} f = \overline{e^{i\theta} g} + e^{-i\theta} h$$

is convex in the direction of the real axis. Hence, by the above lemma, the function

$$e^{-i\theta} h - e^{i\theta} g$$

is a conformal map which is convex in the direction of the real axis. Since $\varphi \in DCP$ the function

$$\varphi * (e^{-i\theta} h - e^{i\theta} g) = e^{-i\theta}(\varphi * h) - e^{i\theta}(\varphi * h) \tag{7}$$

is also convex in the direction of the real axis.

Now, by hypothesis, the function

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h$$

is locally univalent. Hence so is the function

$$e^{-i\theta}(\varphi \,\tilde{*}\, f) = \overline{e^{i\theta}(\varphi * g)} + e^{-i\theta}(\varphi * h).$$

If we take into account that (7) is a conformal map convex in the direction of the real axis, we can apply the above lemma to conclude that

$$e^{-i\theta}(\varphi \,\tilde{*}\, f) = \overline{e^{i\theta}(\varphi * g)} + e^{-i\theta}(\varphi * h)$$

is convex in the direction of the real axis. Hence the function

$$\varphi \,\tilde{*}\, f = \overline{\varphi * g} + \varphi * h \in C_H(\theta).$$

This completes the proof of the theorem.

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