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Approximation of functions belonging to Lip α class by $(N, p_n)(E, 1)$ means of its Fourier series

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Abstract: In this paper, the degree of approximation of a function belonging to Lip α class by $(N, p_n)(E,1)$ means of its Fourier series has been determined.

1. Definitions and Notations

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum s_n is given by $s_n = \sum_{v=0}^{n} u_v$.

If

(1) $$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s \quad \text{as } n \to \infty$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ or $\{s_n\}$ is said to be summable to the definite number s by $(E,1)$ method (Hardy [3]).

Let $\{p_n\}$ be a sequence of real constants such that $p_0 > 0, p_n \geq 0 \ \forall \ n \geq 1$. The sequences to sequence transformation

(2) $$t_n^p = \frac{1}{p_n} \sum_{k=0}^{n} p_{n-k} s_k = \frac{1}{p_n} \sum_{k=0}^{n} p_k s_{n-k}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the coefficients $\{p_n\}$. If $t_n^p \to s$ as $n \to \infty$, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable (N, p_n) to the sum s.

The (N,p_n) transform of the $(E,1)$ transform defines the $(N,p_n)(E,1)$ transform of the partial sum s_n of the series $\sum_{n=0}^{\infty} u_n$.

Thus if

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, E_k^1 = \frac{1}{P_n}\sum_{k=0}^{\infty} p_k\, E_{n-k}^1 \quad \text{as} \quad n \to \infty$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N,p_n)(E,1)$ means or simply, summable $(N,p_n)(E,1)$ to s.

A function $f \in \text{Lip}\,\alpha$ if

(3) $$f(x+t) - f(x) = 0(|t|^{\alpha}), \quad 0 < \alpha \le 1.$$

Let $f(x)$ be periodic with period 2π and Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip α class. The Fourier series of $f(x)$ is given by

(4) $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{n}(a_n \cos nx + b_n \sin nx).$$

The degree of approximation of a function $f: \mathrm{R} \to \mathrm{R}$ by a trigonometric polynomial t_n of order n is defined by

(5) $$\| t_n + f \|_{\infty} = \sup\{ | t_n(x) - f(x) | : x \in \mathrm{R} \}, \quad \text{(Zygmund [9])}.$$

We use the following notations throughout this paper:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

(6) $$N_n(t) = \frac{1}{2\pi P_n}\sum_{k=0}^{n} P_k \frac{\operatorname{Cos}^{n-k}\left(\frac{t}{2}\right)\sin\,(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}}$$

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

2. Theorem: *The degree of approximation of a function f belonging to Lip α by Cesàro means and Nörlund means has been discussed by a number of researchers like Alexits [1], Sahney and Goel [8], Chandra [2], Qureshi [5], Qureshi [6] and Qureshi and Neha [7]. The purpose of this paper is to determine the degree of approximation of a* *function belonging to Lip α class by (N,p_n) (E, 1) product means of its Fourier series in the following form:*

Theorem: *Let (N,p_n) be a regular Nörlund summability method generated by a positive, monotonic decreasing sequence $\{p_n\}$ of real constants. If $f:[-\pi,\pi]\to \mathrm{R}$ is 2π periodic, Lebesgue integrable on $[-\pi,\pi]$ and belonging to Lip α class then the degree of approximation of f by the (N,p_n) (E, 1) means $t_n^{p,E} = \frac{1}{p_n}\sum_{k=0}^{n} p_k E_{n-k}^{1}$ of its Fourier series (4) is given by*

$$\left\| t_n^{NE} - f \right\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^a}\right), 0<a<1 \\ O\left(\frac{\log(n+1)\pi e}{n+1}\right), a=1. \end{cases}$$

3. Lemma: *The proof of the theorem requires following lemmas.*

Lemma 1. *Let $N_n(t)$ be given by (6), then*

$$N_n(t) = 0(n+1)\text{, for } 0<t<\frac{1}{n+1}.$$

Proof:

$$\left|N_n(t)\right| \le \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k \left|\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k(n-k+1)\frac{\left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|}$$

$$= \frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k(n-k+1)$$

$$= (n+1)\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k$$

$$= \left(\frac{n+1}{2\pi}\right)$$

$$= O(n+1)$$

which completes the proof of lemma 1.

Lemma 2. [*Mc Fadden* (4)].

If $\{p_n\}$ is a non-negative and non-increasing sequence, then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n,

$$\left| \sum_{a}^{b} p_k \, e^{i(n-k)t} \right| = O\left(P_\tau\right)$$

where $P_\tau = P_{\left[\frac{1}{t}\right]}$ and $\tau = \left[\frac{1}{t}\right]$.

Lemma 3. $N_n(t) = O\left(\dfrac{P_t}{tP_n}\right)$, for $\dfrac{1}{n+1} < t < \pi$.

Proof: Since, for $\dfrac{1}{n+1} < t < \pi$, $\sin\dfrac{t}{2} > \dfrac{t}{\pi}$, therefore

$$\left| N_n(t) \right| = \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} p_k \frac{\cos^{n-k}\left(\frac{t}{2}\right) \sin\left(n-k+1\right)\frac{t}{2}}{\sin\frac{t}{2}} \right|$$

$$= \frac{1}{2\pi P_n} \left| \frac{I_m \sum_{k=0}^{n} p_k \cos^{n-k}\left(\frac{t}{2}\right) e^{i(n-k+1)\frac{t}{2}}}{\left|\sin\frac{t}{2}\right|} \right|$$

$$\le \frac{1}{2tP_n} \left| \sum_{k=0}^{n} p_k \cos^{n-k}\left(\frac{t}{2}\right) e^{i(n-k)\frac{t}{2}} \right| \left| e^{i\frac{t}{2}} \right|$$

$$\le \frac{1}{2tP_n} \left| \sum_{k=0}^{n} p_k e^{i(n-k)\frac{t}{2}} \right|$$

$$= O\left(\frac{P_\tau}{tP_n}\right), \text{ by lemma (2).}$$

4. Proof of the Theorem: The n^{th} partial sum $S_n(x)$ of the Fourier series (4) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \, dt.$$

Then

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(S_k(x)-f(x))=\frac{1}{2^{n+1}\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\left[\sum_{k=0}^{n}\binom{n}{k}\sin\left(k+\frac{1}{2}\right)t\right]dt.$$

or

$$E_n^1(x)-f(x)=\frac{1}{2^{n+1}\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}(1+e^{it})^n\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}\left(1+\cos t+i\sin t\right)^n\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}I_m\left\{e^{\frac{it}{2}}2^n\cos^n\left(\frac{t}{2}\right)\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)^2\right\}dt$$

$$=\frac{1}{2^{n+1}\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}2^n\cos^n\left(\frac{t}{2}\right)I_m\left(\cos\frac{t}{2}+i\sin\frac{t}{2}\right)\left\{\cos\frac{nt}{2}+i\sin\frac{nt}{2}\right\}dt$$

$$=\frac{1}{2\pi}\int_0^\pi \frac{\phi_x(t)}{\sin\frac{t}{2}}\cos^n\left(\frac{t}{2}\right)\sin(n+1)\frac{t}{2}dt.$$

Now,

$$\frac{1}{P_n}\sum_{k=0}^{n}p_k\left(E_{n-k}^1(x)-f(x)\right)=\int_0^\pi\left[\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_k\frac{\cos^{n-k}\left(\frac{t}{2}\right)\sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}}\right]\phi(t)dt.$$

or,

$$t_n^{NE}(x)-f(x)=\int_0^\pi N_n(t)\,\phi(t)\,dt$$

$$=\int_0^{\frac{1}{n+1}}N_n(t)\,\phi(t)\,dt+\int_{\frac{1}{n+1}}^{\pi}N_n(t)\phi(t)\,dt \tag{7}$$

$$=I_1+I_2 \quad \text{say.}$$

Now,

$$\begin{aligned}\phi(t)&=f(x+t)+f(x-t)-2f(x)\\&=\{f(x+t)-f(x)\}+\{f(x-t)-f(x)\}\\&=O(t^\alpha)+O(t^\alpha)\quad(\because f\in \mathrm{Lip}^\alpha)\\&=O(t^\alpha).\end{aligned}$$

We have,

$$|I_1| \leq \int_0^{\frac{1}{n+1}} \left| N_n(t) \right| \left| \phi(t) \right| dt$$

$$= O(n+1) \int_0^{\frac{1}{n+1}} |\phi(t)| \, dt, \text{ by lemma 1}$$

$$= O(n+1) \int_0^{\frac{1}{n+1}} t^{\alpha} dt$$

$$= O(n+1) \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\frac{1}{n+1}}$$

$$= O(n+1) \left[\frac{1}{(n+1)^{\alpha+1}} \right]$$

$$(8) \qquad = O\left[\frac{1}{(n+1)^{\alpha}} \right].$$

Next,

$$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} \left| N_n(t) \right| \left| \phi(t) \right| dt$$

$$= \int_{\frac{1}{n+1}}^{\pi} O\left(\frac{P_{\tau}}{tP_n} \right) |\phi(t)| \, dt, \text{ by lemma 3}$$

$$= O\left(\frac{1}{P_n} \right) \int_{\frac{1}{n+1}}^{\pi} \frac{P_{\tau}}{t} t^{\alpha} dt$$

$$= O\left(\frac{1}{P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{P_{\tau}}{u^{\alpha+1}} du$$

$$= O\left(\frac{P_{n+1}}{(n+1)P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^{\alpha}} du \left(\because \frac{P_{[u]}}{u} \text{ is decreasing} \right)$$

$$= O\left(\frac{1}{(n+1)} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^{\alpha}} du.$$

$$= O\left(\frac{1}{n+1}\right)\begin{cases}\left(\dfrac{u^{-\alpha+1}}{-\alpha+1}\right)_{\frac{1}{\pi}}^{n+1}, & \alpha \neq 1\\ (\log u)_{\frac{1}{\pi}}^{n+1}, & \alpha = 1\end{cases}$$

(9)
$$= O\left(\frac{1}{n+1}\right)\begin{cases}\dfrac{1}{(1-\alpha)}\left(\dfrac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right), & \alpha \neq 1\\ \log(n+1)\pi, & \alpha = 1.\end{cases}$$

By (7), (8) and (9), we have

$$t_n^{NE}(x) - f(x) = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(n+1)}\right)\left[\dfrac{1}{(1-\alpha)(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right], & \alpha \neq 1\\ O\left(\dfrac{1}{n+1}\right) + O\left(\dfrac{\log(n+1)\pi}{n+1}\right), & \alpha = 1\end{cases}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(1-\alpha)(n+1)^{\alpha}}\right) + O\left(\dfrac{1}{(n+1)^{\alpha}\pi^{1-\alpha}}\right), & \alpha \neq 1\\ O\left(\dfrac{1}{(n+1)}\right) + O\left(\dfrac{\log(n+1)\pi}{n+1}\right), & \alpha = 1\end{cases}$$

$$= \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & 0 < \alpha < 1\\ O\left(\dfrac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1.\end{cases}$$

Then

$$\left\| t_n^{NE} - f \right\|_{\infty} = \sup\left\{ |t_n^{NE}(x) - f(x)| : -\pi \leq x \leq \pi \right\}.$$

or,

$$\left\| t_n^{NE} - f \right\|_{\infty} = \begin{cases} O\left(\dfrac{1}{(n+1)^{\alpha}}\right), & 0 < \alpha < 1\\ O\left(\dfrac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1.\end{cases}$$

Hence the theorem is completely established.

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