

Approximation of functions belonging to Lip α class by $(N, p_n)(E, 1)$ means of its Fourier series

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Abstract: In this paper, the degree of approximation of a function belonging to Lip α class by $(N, p_n)(E, 1)$ means of its Fourier series has been determined.

1. Definitions and Notations

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum s_n is given by $s_n = \sum_{v=0}^n u_v$.

If

$$(1) \quad E_n^1 = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} s_k \rightarrow s \quad \text{as } n \rightarrow \infty$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ or $\{s_n\}$ is said to be summable to the definite number s by $(E, 1)$ method (Hardy [3]).

Let $\{p_n\}$ be a sequence of real constants such that $p_0 > 0, p_n \geq 0 \forall n \geq 1$.

The sequences to sequence transformation

$$(2) \quad t_n^p = \frac{1}{p_n} \sum_{k=0}^n p_{n-k} s_k = \frac{1}{p_n} \sum_{k=0}^n p_k s_{n-k}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the coefficients $\{p_n\}$. If $t_n^p \rightarrow s$ as $n \rightarrow \infty$, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable (N, p_n) to the sum s .

The (N, p_n) transform of the $(E, 1)$ transform defines the $(N, p_n)(E, 1)$ transform of the partial sum s_n of the series $\sum_{n=0}^{\infty} u_n$.

Thus if

$$t_n^{NE} = \frac{1}{P_n} \sum_{k=0}^n p_{n-k} E_k^1 = \frac{1}{P_n} \sum_{k=0}^{\infty} p_k E_{n-k}^1 \text{ as } n \rightarrow \infty$$

then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N, p_n)(E, 1)$ means or simply,

summable $(N, p_n)(E, 1)$ to s .

A function $f \in \text{Lip } \alpha$ if

$$(3) \quad |f(x+t) - f(x)| = O(|t|^\alpha), \quad 0 < \alpha \leq 1.$$

Let $f(x)$ be periodic with period 2π and Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip α class. The Fourier series of $f(x)$ is given by

$$(4) \quad f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The degree of approximation of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ by a trigonometric polynomial t_n of order n is defined by

$$(5) \quad \|t_n + f\|_{\infty} = \sup \{ |t_n(x) - f(x)| : x \in \mathbb{R} \}, \quad (\text{Zygmund [9]}).$$

We use the following notations throughout this paper:

$$(6) \quad \begin{aligned} \phi(t) &= f(x+t) + f(x-t) - 2f(x), \\ N_n(t) &= \frac{1}{2\pi P_n} \sum_{k=0}^n P_k \frac{\text{Cos}^{n-k} \left(\frac{t}{2} \right) \sin \left(n-k+1 \right) \frac{t}{2}}{\sin \frac{t}{2}} \end{aligned}$$

$$\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t} \right].$$

2. Theorem: *The degree of approximation of a function f belonging to Lip α by Cesàro means and Nörlund means has been discussed by a number of researchers like Alexits [1], Sahney and Goel [8], Chandra [2], Qureshi [5], Qureshi [6] and Qureshi and Neha [7]. The purpose of this paper is to determine the degree of approximation of a*

function belonging to Lip α class by $(N, p_n)(E, 1)$ product means of its Fourier series in the following form:

Theorem: Let (N, p_n) be a regular Nörlund summability method generated by a positive, monotonic decreasing sequence $\{p_n\}$ of real constants. If $f: [-\pi, \pi] \rightarrow \mathbb{R}$ is 2π periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip α class then the degree of approximation of f by the $(N, p_n)(E, 1)$ means $t_n^{p, E} = \frac{1}{p_n} \sum_{k=0}^n p_k E_{n-k}^1$ of its Fourier series (4) is given by

$$\|t_n^{NE} - f\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^a}\right), & 0 < a < 1 \\ O\left(\frac{\log(n+1)\pi e}{n+1}\right), & a = 1. \end{cases}$$

3. Lemma: The proof of the theorem requires following lemmas.

Lemma 1. Let $N_n(t)$ be given by (6), then

$$N_n(t) = 0(n+1), \text{ for } 0 < t < \frac{1}{n+1}.$$

Proof:
$$|N_n(t)| \leq \frac{1}{2\pi P_n} \sum_{k=0}^n p_k \left| \frac{\cos^{n-k}\left(\frac{t}{2}\right) \sin(n-k+1)\frac{t}{2}}{\sin\frac{t}{2}} \right|$$

$$\leq \frac{1}{2\pi P_n} \sum_{k=0}^n p_k (n-k+1) \frac{\left|\sin\frac{t}{2}\right|}{\left|\sin\frac{t}{2}\right|}$$

$$= \frac{1}{2\pi P_n} \sum_{k=0}^n p_k (n-k+1)$$

$$= (n+1) \frac{1}{2\pi P_n} \sum_{k=0}^n p_k$$

$$= \left(\frac{n+1}{2\pi}\right)$$

$$= O(n+1)$$

which completes the proof of lemma 1.

Lemma 2. [Mc Fadden (4)].

If $\{p_n\}$ is a non-negative and non-increasing sequence, then for $0 \leq a < b \leq \infty$, $0 \leq t \leq \pi$ and for any n ,

$$\left| \sum_a^b p_k e^{i(n-k)t} \right| = O(P_\tau)$$

where $P_\tau = P \left[\frac{1}{t} \right]$ and $\tau = \left[\frac{1}{t} \right]$.

Lemma 3. $N_n(t) = O \left(\frac{P_t}{tP_n} \right)$, for $\frac{1}{n+1} < t < \pi$.

Proof: Since, for $\frac{1}{n+1} < t < \pi$, $\sin \frac{t}{2} > \frac{t}{\pi}$, therefore

$$\begin{aligned} |N_n(t)| &= \frac{1}{2\pi P_n} \left| \sum_{k=0}^n p_k \frac{\cos^{n-k} \left(\frac{t}{2} \right) \sin (n-k+1) \frac{t}{2}}{\sin \frac{t}{2}} \right| \\ &= \frac{1}{2\pi P_n} \left| \frac{I_m \sum_{k=0}^n p_k \cos^{n-k} \left(\frac{t}{2} \right) e^{i(n-k+1) \frac{t}{2}}}{\left| \sin \frac{t}{2} \right|} \right| \\ &\leq \frac{1}{2tP_n} \left| \sum_{k=0}^n p_k \cos^{n-k} \left(\frac{t}{2} \right) e^{i(n-k) \frac{t}{2}} \right| e^{i \frac{t}{2}} \\ &\leq \frac{1}{2tP_n} \left| \sum_{k=0}^n p_k e^{i(n-k) \frac{t}{2}} \right| \\ &= O \left(\frac{P_\tau}{tP_n} \right), \text{ by lemma (2).} \end{aligned}$$

4. Proof of the Theorem: The n^{th} partial sum $S_n(x)$ of the Fourier series (4) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi_x(t) \frac{\sin \left(n + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dt.$$

Then

$$\frac{1}{2^n} \sum_{k=0}^n$$

or

$$E_n^1(x)$$

Now,

$$\frac{1}{P_n} \sum_{k=0}^n$$

or,

(7)

Now,

Then

$$\frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (S_k(x) - f(x)) = \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} \left[\sum_{k=0}^n \binom{n}{k} \sin \left(k + \frac{1}{2} \right) t \right] dt.$$

or

$$\begin{aligned} E_n^1(x) - f(x) &= \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} I_n \left[e^{\frac{it}{2}} (1 + e^{it})^n \right] dt \\ &= \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} I_n \left[e^{\frac{it}{2}} (1 + \cos t + i \sin t)^n \right] dt \\ &= \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} I_n \left[e^{\frac{it}{2}} 2^n \cos^n \left(\frac{t}{2} \right) \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right)^2 \right] dt \\ &= \frac{1}{2^{n+1}\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} 2^n \cos^n \left(\frac{t}{2} \right) I_n \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right) \left\{ \cos \frac{nt}{2} + i \sin \frac{nt}{2} \right\} dt \\ &= \frac{1}{2\pi} \int_0^\pi \frac{\phi_x(t)}{\sin \frac{t}{2}} \cos^n \left(\frac{t}{2} \right) \sin(n+1) \frac{t}{2} dt. \end{aligned}$$

Now,

$$\frac{1}{P_n} \sum_{k=0}^n p_k (E_{n-k}^1(x) - f(x)) = \int_0^\pi \left[\frac{1}{2^n P_n} \sum_{k=0}^n p_k \frac{\cos^{n-k} \left(\frac{t}{2} \right) \sin(n-k+1) \frac{t}{2}}{\sin \frac{t}{2}} \right] \phi(t) dt.$$

or,

$$t_n^{NE}(x) - f(x) = \int_0^\pi N_n(t) \phi(t) dt.$$

(7)

$$\begin{aligned} &= \int_0^{\frac{1}{n+1}} N_n(t) \phi(t) dt + \int_{\frac{1}{n+1}}^\pi N_n(t) \phi(t) dt \\ &= I_1 + I_2 \text{ say.} \end{aligned}$$

Now,

$$\begin{aligned} \phi(t) &= f(x+t) + f(x-t) - 2f(x) \\ &= \{f(x+t) - f(x)\} + \{f(x-t) - f(x)\} \\ &= O(t^\alpha) + O(t^\alpha) \quad (\because f \in \text{Lip}^\alpha) \\ &= O(t^\alpha). \end{aligned}$$

$0 \leq t \leq \pi$

(4) is given

We have,

$$\begin{aligned}
 |I_1| &\leq \int_0^{\frac{1}{n+1}} |N_n(t)| |\phi(t)| dt \\
 &= O(n+1) \int_0^{\frac{1}{n+1}} |\phi(t)| dt, \text{ by lemma 1} \\
 &= O(n+1) \int_0^{\frac{1}{n+1}} t^\alpha dt \\
 &= O(n+1) \left[\frac{t^{\alpha+1}}{\alpha+1} \right]_0^{\frac{1}{n+1}} \\
 &= O(n+1) \left[\frac{1}{(n+1)^{\alpha+1}} \right] \\
 (8) \quad &= O \left[\frac{1}{(n+1)^\alpha} \right].
 \end{aligned}$$

Next,

$$\begin{aligned}
 |I_2| &\leq \int_{\frac{1}{n+1}}^{\pi} |N_n(t)| |\phi(t)| dt \\
 &= \int_{\frac{1}{n+1}}^{\pi} O \left(\frac{P_\tau}{t P_n} \right) |\phi(t)| dt, \text{ by lemma 3} \\
 &= O \left(\frac{1}{P_n} \right) \int_{\frac{1}{n+1}}^{\pi} \frac{P_\tau}{t} t^\alpha dt \\
 &= O \left(\frac{1}{P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{P_\tau}{u^{\alpha+1}} du \\
 &= O \left(\frac{P_{n+1}}{(n+1)P_n} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^\alpha} du \left(\because \frac{P[u]}{u} \text{ is decreasing} \right) \\
 &= O \left(\frac{1}{(n+1)} \right) \int_{\frac{1}{\pi}}^{n+1} \frac{1}{u^\alpha} du.
 \end{aligned}$$

(9)

By (7), (8)

Then

or,

Hence the t

$$\begin{aligned}
 &= O\left(\frac{1}{n+1}\right) \begin{cases} \left(\frac{u^{-\alpha+1}}{-\alpha+1}\right)^{\frac{1}{\pi}}, & \alpha \neq 1 \\ (\log u)^{\frac{1}{\pi}}, & \alpha = 1 \end{cases} \\
 (9) \quad &= O\left(\frac{1}{n+1}\right) \begin{cases} \frac{1}{(1-\alpha)} \left(\frac{1}{(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right), & \alpha \neq 1 \\ \log(n+1)\pi, & \alpha = 1. \end{cases}
 \end{aligned}$$

By (7), (8) and (9), we have

$$\begin{aligned}
 t_n^{NE}(x) - f(x) &= \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right) + O\left(\frac{1}{(n+1)}\right) \left[\frac{1}{(1-\alpha)(n+1)^{\alpha-1}} - \pi^{\alpha-1}\right], & \alpha \neq 1 \\ O\left(\frac{1}{n+1}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right), & \alpha = 1 \end{cases} \\
 &= \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right) + O\left(\frac{1}{(1-\alpha)(n+1)^\alpha}\right) + O\left(\frac{1}{(n+1)^\alpha \pi^{1-\alpha}}\right), & \alpha \neq 1 \\ O\left(\frac{1}{(n+1)}\right) + O\left(\frac{\log(n+1)\pi}{n+1}\right), & \alpha = 1 \end{cases} \\
 &= \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1. \end{cases}
 \end{aligned}$$

Then

$$\left\| t_n^{NE} - f \right\|_\infty = \sup \{ |t_n^{NE}(x) - f(x)| : -\pi \leq x \leq \pi \}.$$

or,

$$\left\| t_n^{NE} - f \right\|_\infty = \begin{cases} O\left(\frac{1}{(n+1)^\alpha}\right), & 0 < \alpha < 1 \\ O\left(\frac{\log(n+1)\pi e}{n+1}\right), & \alpha = 1. \end{cases}$$

Hence the theorem is completely established.

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Abstract: In this paper we study that such a manifold with a vector field ξ and the vector field η is called a Riemannian manifold with a vector field ξ and the vector field η .

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Key words and phrases: sectional curvature, vector field, Riemannian manifold, covariant derivative, divergence, Lie derivative, Killing vector field, conformal vector field, homothetic vector field, infinitesimal isometry, infinitesimal conformal transformation, infinitesimal homothetic transformation, infinitesimal isometry of the second kind, infinitesimal conformal transformation of the second kind, infinitesimal homothetic transformation of the second kind.

1. Introduction

In (1989) K. I. Mihai and R. Sudipta Biswas introduced the concept of a Riemannian manifold with a vector field ξ and the vector field η .

In this paper we study that such a manifold with a vector field ξ and the vector field η is called a Riemannian manifold with a vector field ξ and the vector field η .

2. Preliminaries

A differentiable manifold M is called a Riemannian manifold with a vector field ξ and the vector field η if it admits a covariant vector field ω and the vector field η .