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Approximation of the conjugate of a function belonging to Lip (α,p) class by matrix means

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Abstract: The present paper deals with approximation of the conjugate of a function belonging to the $\mathrm{Lip}(\alpha, p)$ class by matrix summability method. A new estimate on the degree of approximation of conjugate function $\bar{f}$, conjugate to a function $f \in \mathrm{Lip}(\alpha, p)$, has been determined by matrix summability of conjugate series of a Fourier series.

Key words and phrases: Degree of approximation, matrix summability, Fourier series conjugate series of the Fourier series, $\mathrm{Lip}(\alpha, p)$ class.

Subject classification: 42B05, 42B08

§ 1. Let f be 2π-periodic, integrable over $(-\pi, \pi)$ in the sense of Legesgue, then its Fourier series is given by

$$f(t) \sim \frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_o + \sum_{n=1}^{\infty} A_n(t) \tag{1}$$

with partial sum $S_n(x)$.

The conjugate series of the Fourier series (1) given by

$$\sum_{n=1}^{\infty} (a_n \sin nt - \cos nt) = -\sum_{n=1}^{\infty} B_n(t) \tag{2}$$

with partial sum $S_n(x)$.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying the Silverman-Tőeplitz [5] conditions of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \text{ as } n \to \infty,$$

$$a_{n,k} = 0, \text{ for } k > n$$

and $\sum_{k=0}^{n} |a_{n,k}| \leq M$, where, M is a finite positive constant.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum is given by

$$S_n = \sum_{v=0}^{\infty} u_v.$$

This sequence–to-sequence transformation

$$t_n = \sum_{k=1}^{n} a_{n,n-k} \, S_{n-k} \tag{3}$$

defines the sequence $\{t_n\}$ of matrix means of sequence $\{S_n\}$, generated by the sequence of coefficient $\{a_{n,k}\}$. If

$$t_n \to S \text{ as } n \to \infty,$$

then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{S_n\}$ is said to be summable by matrix method (T) to S. It is dentoed by

$$t_n \to S(T) \text{ as } n \to \infty \text{ (Zygmund [6])}.$$

The summability method (T) reduces to

(i) Harmonic means, when $a_{n,k} = \dfrac{1}{(n-k+1)\log n} \; \forall \; 0 \leq k \leq n$.

(ii) (H,p) means, when $a_{n,k} = \dfrac{1}{\log^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^q (k+1)$.

(iii) (N, p_n), when $a_{n,k} = \dfrac{p_{n-k}}{p_n}$, where $p_n = \sum_{k=0}^{n} p_k \neq 0$.

(iv) (N,p,q) means, when $a_{n,k} = \dfrac{p_{n-k} q_k}{R_n}$, where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$.

The L^p norms is defined by

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p}, \; p \geq 1$$

and the degree of approximation $E_n(f)$ under norm $\| \; \|_p$ is given by (Zygmund [6])

$$E_n(f) = \min_{t_n} \| t_n(x) - f(x) \|_p ,$$

$t_n(x)$ where $t_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in \text{Lip}\, \alpha$, if

$$|f(x+t)-f(x)| = O(|t|^{\alpha}), \text{ for } 0<\alpha\le 1 \text{ and}$$

$f \in \text{Lip}(\alpha, p)$, for $0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} \left|f(x+t)-f(x)\right|^p dx\right)^{\frac{1}{p}} = O(|t|^{\alpha}),\ 0<\alpha\le 1,\ p\ge 1.$$

We write

(4) $$\psi(t) = f(x+t) - f(x-t)$$

(5) $$\bar{K}(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}$$

(6) $$A_{n,n} = \sum_{k=0}^{n} a_{n,n-k} = O(1)$$

(7) $$A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$$

$\tau = \left[\frac{1}{t}\right]$, where, τ denotes the greatest integer not greater then $\frac{1}{t}$.

§2 Bernstin [1], used $(C,1)$ means to obtain the degree of approximation of lip 1 function. Jackson [2] determined the degree of approximation by using (C,δ) method in lip α class, for $0<\alpha<1$. First time, the concept of the degree of approximation of the conjugate function $\overline{f}(x)$ has been introduced by Qureshi [4]. He used Lip α class functions by Nörlund method. The purpose of this paper is to obtain the approximation of $f(x)$, the conjugate of a function f belonging to $\text{Lip}(\alpha,p)$ class, by matrix means of conjugate series of a Fourier series. In fact, I prove following theorem:

Theorem: *$T=(a_{n,k})$ be an infinite lower regular triangular matrix such that the element $(a_{n,k})$ be non-negative, non-decreasing with $k\le n$, then the degree of approximation of function $\overline{f}(x)$, conjugate to a 2π-periodic function $f(x)\in Lip(\alpha,p)$, $0<\alpha\le 1, p\ge 1$, by matrix means (T) of its conjugate series (2), is given by*

(8) $$\|\overline{t}_n(x) - \overline{f}(x)\|_p = O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)$$

where $\overline{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k}\,\overline{S}_{n-k}(x)$ is the matrix means of the series (2).

§ 3. For the proof of my theorem following lemmas are required.

Lemma 1: Let $\bar{K}(n,t)$ be given in (5), then

$$\bar{K}(n,t) = O\left(\frac{1}{t}\right), \text{ for } 0 \le t < \frac{1}{n}.$$

Proof:

$$|\bar{K}(n,t)| = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\left|\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2t}\sum_{k=0}^{n} a_{n,n-k}$$

$$= \frac{A_{n,n}}{2t}$$

$$= O\left(\frac{1}{t}\right)O(1)$$

$$= O\left(\frac{1}{t}\right).$$

Lemma 2: (Lal [3]), *If $a_{n,k}$ is non-negative and non-decreasing with $k \le n$ then for, $0 \le a < b \le \infty$, $0 \le t \le \pi$ and for any n.*

We have

$$\sum_{k=a}^{b}\left|a_{n,n-k}e^{i(n-k)t}\right| = O(A_{n,\tau}) \text{ where } \tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right].$$

Lemma 3: *Let $\bar{K}(n,t)$ be given in (5) and under the condition of my theorem on $(a_{n,k})$, we have*

$$\bar{K}(n,t) = O\left(\frac{A_{n,\tau}}{t}\right), \text{ for } \frac{1}{n} < t \le \pi.$$

Proof: It is well known that, $\sin\frac{t}{2} \ge \frac{t}{\pi}$ (since, $\sin\theta \ge \frac{2\theta}{\pi}, 0 < \theta < \pi$, Jordan's Lemma).

Now, for $t > 0$ and $\tau \le n$, we have

$$|\bar{K}(n,t)| = \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$= \left|\frac{1}{2\pi\sin\frac{t}{2}} \text{ real part of } \sum_{k=0}^{n} a_{n,n-k}\, e^{i\left(n-k+\frac{1}{2}\right)t}\right|$$

$$\le \left|\frac{1}{2t} \text{ real part of } \sum_{k=0}^{n} a_{n,n-k}\, e^{i(n-k)t}\, e^{\frac{it}{2}}\right|$$

$$\leq \left| \frac{1}{2t} \sum_{k=0}^{n} a_{n,n-k}\ e^{i(n-k)t}\ | e^{\frac{it}{2}} | \right|$$

$$= O\left(\frac{1}{t}\right) \left| \sum_{k=0}^{n} a_{n,n-k}\ e^{i(n-k)t} \right| \qquad \left(\Theta \mid e^{\frac{it}{2}} \mid \leq 1\right)$$

$$= O\left(\frac{A_{n,\tau}}{t}\right) \text{ by Lemma 2.}$$

§ 4. n^{th} partial sum $\overline{S}(x)$ of the series (2) is given by

$$\overline{s}_n(x) - \overline{f}(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\psi(t) \cos\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}}\, dt$$

then,

$$\sum_{k=0}^{n} a_{n,n-k} \left(\overline{S}_{n-k}(x) - \overline{f}(x)\right) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos\left(n - k + \frac{1}{2}\right)t}{\sin \frac{t}{2}}\, dt$$

or,

$$\overline{t}_n(x) - \overline{f}(x) = \int_0^{\pi} \psi(t)\, \overline{K}(n,t)\, dt$$

$$= \int_0^{\frac{1}{n}} \psi(t)\, \overline{K}(n,t)\, dt + \int_{\frac{1}{n}}^{\pi} \psi(t)\, \overline{K}(n,t)\, dt$$

$$= I_1 + I_2, \text{ say.} \tag{9}$$

Applying Hölder's inequality, Lemma 1 and fact that $\psi(t) \in \text{Lip}(\alpha, p)$, we have

$$| I_1 | \leq \left[\int_0^{\frac{1}{n}} \left| \frac{t\psi(t)}{t^{\alpha}} \right|^p dt \right]^{\frac{1}{p}} \left[\int_0^{\frac{1}{n}} \left| \frac{\overline{K}(n,t)}{t^{1-\alpha}} \right|^q dt \right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_0^{\frac{1}{n}} t^{p-1} dt \right)^{\frac{1}{p}} \right] O\left[\left(\int_0^{\frac{1}{n}} t^{(\alpha-2)q} dt \right)^{\frac{1}{q}} \right]$$

$$= O\left[\left\{ \left(\frac{t^p}{p} \right)_0^{\frac{1}{n}} \right\}^{\frac{1}{p}} \right] O\left[\left\{ \left[\frac{t^{(\alpha-2)q+1}}{(\alpha-2)q+1} \right]_0^{\frac{1}{n}} \right\}^{\frac{1}{q}} \right]$$

$$= O\left(\frac{1}{n}\right) O\left(\frac{1}{n^{\alpha-2+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right). \tag{10}$$

Using Hölder's inequality and Lemma 3, we have

$$|I_2| \leq \left[\int_{\frac{1}{n}}^{\pi} \left|\frac{t^{-\delta}\psi(t)}{t^{\alpha}}\right|^p dt\right]^{\frac{1}{p}} \left[\int_{\frac{1}{n}}^{\pi} \left|\frac{\overline{K}(t)}{t^{-\delta-\alpha}}\right|^q dt\right]^{\frac{1}{q}}$$

$$= O\left[\left(\int_{\frac{1}{n}}^{\pi} \left(t^{-\frac{1}{p}-\delta}\right)^p dt\right)^{\frac{1}{p}}\right] O\left[\left(\int_{\frac{1}{n}}^{\pi} \left(\frac{A_{n,\tau}}{t^{1-\delta-\alpha}}\right)^q dt\right)^{\frac{1}{q}}\right]$$

$$= O\left[\left\{\left(\frac{t^{-\delta p}}{-\delta p}\right)_{\frac{1}{n}}^{\pi}\right\}^{\frac{1}{p}}\right] O(A_{n,n})\, O\left[\left\{\left(\frac{t^{(\delta+\alpha-1)q+1}}{(\delta+\alpha-1)q+1}\right)_{\frac{1}{n}}^{\pi}\right\}^{\frac{1}{q}}\right]$$

$$= O(n^{\delta})\, O\left(\frac{1}{n^{\delta+\alpha-1+\frac{1}{q}}}\right)$$

$$= O\left(\frac{1}{n^{\alpha-1+\frac{1}{q}}}\right)$$

(11) $$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$ and q is the conjugate index of p. Combining the conditions (9)-(11), we have

$$\left|\overline{t}_n(x)-\overline{f}(x)\right| = O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right).$$

Now, $$\left\|\overline{t}_n(x)-\overline{f}(x)\right\|_p = \left[\int_0^{2\pi}\left|\overline{t}_n(x)-\overline{f}(x)\right|^p dx\right]^{\frac{1}{p}}$$

$$= O\left[\int_0^{2\pi}\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)^p dx\right]^{\frac{1}{p}}$$

$$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right)\left[\int_0^{2\pi} dx\right]^{\frac{1}{p}}$$

(9) $$= O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right).$$

This completes the proof of the theorem.

§ 5. Following corollary can be derived from the main theorem.

Corollary 1. *If $p \to \infty$, the degree of approximation of $\overline{f}(x)$, the conjugate of a function $f \in \operatorname{Lip}\alpha$ class by matrix means is given by*

$$\left\| \overline{t}_n(x) - \overline{f}(x) \right\|_\infty = \sup_{0 \le x \le 2\pi} | \overline{t}_n(x) - \overline{f}(x) | = O\left(\frac{1}{n^\alpha}\right), \quad \text{for } 0 < \alpha < 1,$$

where, $\overline{t}_n(x) = \sum_{k=0}^{n} a_{n,n-k} \overline{S}_{n-k}(x)$ is the matrix (T) means of the series (2).

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