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Approximation of the Lip ($\xi(t)$, p) Class Functions by Matrix-cesáro Summability Method

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Abstract. The degree of approximation of functions belonging to Lipα, Lip(α,p) and Lip($\xi(t)$, p) class by Cesàro, Nörlund, Euler and matrix summability method has been determined by number of researchers of Modern Analysis. Most of the summability methods are derived from matrix method. In this paper, I have taken product of two summability methods, matrix and Cesàro; and established a new theorem on the degree of approximation of the function f belonging to Lip ($\xi(t)$, p) class by matrix- Cesàro method.

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1. INTRODUCTION

Bernstin [3], used (C,1) means to obtain the degree of approximation function f by lip 1 class. Jackson [6] determined the degree of approximation by using (C,δ) method in Lip α class, for $0<\alpha<1$. Results of Alexits [2], Chandra [4], Sahney & Goel [14], Sahney & Rao [15], Alexits & Leindler [1] for the degree of approximation of functions $f \in Lip\alpha$ are not satisfied for n= 0, 1 or $\alpha = 1$. Above mentioned results have been generalized by number of researchers like Khan [7], Qureshi [10, 11, 12 & 13], Lal & Nigam [8], Lal & Singh [9] and Dhakal [5]; and determined the degree of approximation of a function f belonging to Lip α, Lip (α, p) and Lip $(\xi(t), p)$ by using Cesàro, Nörlund, generalized Nörlund, Riesz, matrix and (C,1)(E,1) summability method. But till now no work seems to have been done to obtain the degree of approximation of functions by product summability of matrix means and Cesàro means of order one i.e. $T(C_1)$. In an attempt to make an advanced study in this direction, in this paper, a new theorems on the approximation of function $f \in$ Lip $(\xi(t),p)$ class has been established.

2. DEFINITIONS AND NOTATIONS

Let f be 2π-periodic, integrable over $(-\pi,\pi)$ in the sense of Lebesgue, then its Fourier series is given by $f(t) \sim \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$

(1)

with partial sum $S_n(x)$.

The L^p norm is defined by $\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p dx \right)^{\frac{1}{p}}$, $p \geq 1$

and the degree of approximation $E_n(f)$ under norm $\| \|_p$ is given by(Zygmund [17])

$$E_n(f) = \min \|T_n - f\|_p,$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^{\alpha})$, for $0 < \alpha \leq 1$.

$f \in \mathrm{Lip}\,(\alpha, p)$, for $0 \le x \le 2\pi$, if $\left(\int_0^{2\pi} | f(x+t) - f(x) |^p dx \right)^{\frac{1}{p}} = O\left(|t|^\alpha\right)$, $0 < \alpha \le 1$,

$p \ge 1$.

Given a positive increasing function $\xi(t)$, $p \ge 1$, $f \in \mathrm{Lip}\,(\xi(t), p)$ if

$$\left(\int_0^{2\pi} \left| \left(f(x+t) - f(x)\right) \right|^p dx \right)^{\frac{1}{p}} = O\left(\xi(t)\right).$$

It is observed that $\mathrm{Lip}\,(\xi(t), p) \xrightarrow{\xi(t)=t^\alpha} \mathrm{Lip}(\alpha, p) \xrightarrow{p \to \infty} \mathrm{Lip}\alpha$.

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose nth partial sum is given by $S_n = \sum_{ki=0}^{n} u_k$.

Cesàro means (C, 1) of sequence $\{S_n\}$ is given by $\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k$.

If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesàro means (C,1) to S.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying the Silverman-Töeplitz [16] conditions of regularity i.e.

$\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$, $a_{n,k} = 0$, for $k > n$ and $\sum_{k=0}^{n} \left| a_{n,k} \right| \le M$, a finite constant.

Matrix- Cesàro means T (C_1) of the sequence $\{S_n\}$ is given by

$$t_n = \sum_{k=0}^{n} a_{n,n-k}\, \sigma_{n-k} = \sum_{k=0}^{n} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=0}^{n-k} S_r .$$

If $t_n \to S$ as $n \to \infty$, then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by matrix- Cesàro means T (C_1) method to S.

Important particular cases of matrix- Cesàro means are:

(i) $(N,p_n)C_1$ means, when $a_{n,n-k} = \frac{p_k}{P_n}$, where $P_n = \sum_{k=0}^{n} p_k \neq 0$.

(ii) $(\tilde{N},p_n)C_1$ means, when $a_{n,n-k} = \frac{p_{n-k}}{P_n}$

(iii) (N, p, q) C_1 means, $a_{n,n-k} = \frac{p_k\, q_{n-k}}{R_n}$, where $R_n = \sum_{k=0}^{n} p_k\, q_{n-k} \neq 0$.

We write

$$\phi(t) = f(x+t) + f(x-t) - f(x) \tag{2}$$

$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{(n-k+1)} \frac{\sin^2 (n-k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}} \tag{3}$$

3. THEOREM

Quite a good amount of works are known for the degree of approximation of the function $f \in Lip\alpha$, Lip (α, p) and Lip $(\xi(t), p)$ class by various summability methods. The purpose of the present paper is to obtain the degree of approximation of a function $f \in$ Lip $(\xi(t), p)$ class by matrix- Cesàro T (C_1) summability method. We prove the following theorem:

Theorem. Let $f: R \to R$ is 2π-periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to Lip$(\xi(t), p)$ class then the degree of approximation of f matrix-Cesàro means of Fourier series (1) is given

$$\| t_n - f \|_p = O\left((n+1)^{\frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right),$$

(3)

provided $T = (a_{n,k})$ be an infinite lower triangular matrix whose elements $(a_{n,k})$ positive and monotonic increasing in k with $0 \le k \le n$ such that

$$\sum_{k=0}^{n} a_{n,n-k} = 1 \text{ and}$$

(4)

$$\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} = O\left(\frac{1}{n+1}\right). \tag{5}$$

And $\xi(t)$ satisfies the following conditions:

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is monotonic decreasing} \tag{6}$$

$$\left[\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right), \tag{7}$$

$$\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\phi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left((n+1)^{\delta}\right) \tag{8}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$, q the conjugate index of p and conditions (7) & (8) hold uniformly in x.

4. LEMMAS

Lemma 1: For $0 < t < \frac{1}{n+1}$ and fact that $\frac{1}{\sin t} \le \frac{\pi}{2t}$ for $0 < t \le \frac{\pi}{2}$,

$$K(n,t) = O(n+1). \tag{9}$$

Proof: $K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2 (n-k+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}$

$$= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}\,(n-k+1)$$

$\left(\because \sin n\theta \le n \sin\theta \le n\theta \text{ for } 0 < \theta < \frac{1}{n}\right)$

$$\leq \frac{n+1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}$$

$$= \frac{n+1}{2\pi}$$

$$= O(n+1).$$

Lemma 2: For $\frac{1}{n+1} < t < \pi$

$$K(n,t) = O\left(\frac{1}{(n+1)t^2}\right). \tag{10}$$

Proof:
$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\pi^2}{t^2}, \text{ by Jordan's Lemma}$$

$$= \frac{\pi}{2t^2} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1}$$

$$= \frac{\pi}{2t^2} O\left(\frac{1}{n+1}\right), \text{ from condition (5).}$$

$$= O\left(\frac{1}{(n+1)t^2}\right).$$

5. PROOF OF THE THEOREM

n^{th} partial sum $S_n(x)$ of the Fourier series (1) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dt$$

The (C,1) transform i.e. σ_n of S_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(S_k(x) - f(x)\right) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n}\sin\left(k+\tfrac{1}{2}\right)t\,dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\phi(t)\,\frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

The matrix means of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,k}\left(\sigma_k(x) - f(x)\right) = \int_0^{\pi}\phi(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(k+1)}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

or $$\sum_{k=0}^{n} a_{n,n-k}\left(\sigma_{n-k}(x) - f(x)\right) = \int_0^{\pi}\phi(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$t_n(x) - f(x) = \int_0^{\pi}\phi(t)\,K(n,t)\,dt$$

$$= \int_0^{\frac{1}{n+1}}\phi(t)\,K(n,t)\,dt + \int_{\frac{1}{n+1}}^{\pi}\phi(t)\,K(n,t)\,dt$$

$$= J_1 + J_2,\ \text{say.} \tag{11}$$

Applying Hölder inequality, Lemma 1 and fact that $\phi(t) \in Lip(\xi(t), p)$, we have,

$$|J_1| \le \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)|K(n,t)|}{t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left\{\int_{\epsilon}^{\frac{1}{n+1}} t^{-q}dt\right\}^{\frac{1}{q}}$$, for some $0 < \epsilon < \frac{1}{n+1}$, by second mean value theorem for integrals.

$$= O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\left(\frac{t^{-q+1}}{-q+1}\right)_{\epsilon}^{\frac{1}{n+1}}\right]^{\frac{1}{q}}$$

$$= O\left((n+1)^{1-\frac{1}{q}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

$$= O\left((n+1)^{\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right). \tag{12}$$

Similarly for the J_2, we have

$$|J_2| \le \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)|K(n,t)|}{t^{-\delta}}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+2}(n+1)}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= \left((n+1)^{\delta}\,\xi\left(\tfrac{1}{n+1}\right)\right)\left\{\int_{\frac{1}{n+1}}^{\pi} t^{-q(1-\delta)}\,dt\right\}^{\frac{1}{q}} \qquad \text{by condition (6)}$$

$$= O\left((n+1)^{\delta}\,\xi\left(\tfrac{1}{n+1}\right)\right)\left\{\left(\frac{t^{-q(1-\delta)+1}}{-q(1-\delta)+1}\right)_{\frac{1}{n+1}}^{\pi}\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\,\xi\left(\tfrac{1}{n+1}\right)\right)\,(n+1)^{1-\delta-\frac{1}{q}}$$

$$= O\left((n+1)^{1-\frac{1}{q}}\,\xi\left(\tfrac{1}{n+1}\right)\right)$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right).$$

(13)

By (11), (12) and (13), we have

$$|t_n - f| = \left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right)$$

or

$$\|t_n - f\|_p = O\left\{ \int_0^{2\pi} \left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right)^p dx \right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right) \left\{ \int_0^{2\pi} dx \right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\frac{1}{p}} \xi(\tfrac{1}{n+1}) \right). \tag{14}$$

6. APPLICATIONS

Following corollaries may be derided from the theorem.

Corollary 1. If $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, then the degree of approximation of a function

$f \in \mathrm{Lip}\,(\alpha, p)$, $\frac{1}{p} < \alpha \leq 1$, is given by

$$\|t_n - f\|_p = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}} \right).$$

Corollary 2. If $p \to \infty$ in corollary 1, then the degree of approximation of a function $f \in \mathrm{Lip}\,\alpha$, for $0 < \alpha \leq 1$, is

$$\| t_n - f \|_\infty = \begin{cases} O\left(\dfrac{1}{(n+1)^\alpha}\right), & \text{for } 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)\pi e}{(n+1)}\right), & \text{for } \alpha = 1. \end{cases}$$

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