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Associated polynomials to Dirichlet and Fejér kernels

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Abstract: We show that the Fejér kernel generates the fifth-kind Chebyshev polynomials.

Key words: Kernels in Fourier series; Chebyshev polynomials

Introduction

In the original approach to Fourier series, it is convenient to consider the following partial sums for the interval $[-\pi,\pi]$:

$$(1)\qquad f_n(y) = \tfrac{1}{2}a_0 + a_1\cos y + \ldots + a_n\cos(ny) + b_1\sin(y) + \ldots + b_n\sin(ny),$$

assuming for a_r, b_r the values:

$$(2)\qquad a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt,\ \ b_r = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(rt)\,dt,$$

and investigate what happens if n increases to infinity. From (1) and (2) we obtain:

$$(3)\qquad f_n(y) = \int_{-\pi}^{\pi} f(t)\,\underset{D}{K_n}(t-y)\,dt,$$

with the Dirichlet kernel [1–3]:

$$(4)\qquad \underset{D}{K_n}(t-y) = \frac{1}{2\pi}\,\frac{\sin\left[\left(n+\frac{1}{2}\right)(t-y)\right]}{\sin\left(\frac{t-y}{2}\right)}$$

Then we hope that with n increasing to infinity, $f_n(y)$ approaches $f(y)$ with an error which can be made arbitrarily small. This requires a very strong focusing power of $\underset{D}{K_n}(t-y)$, that is, we would like to have the strict property:

$$(5)\qquad \lim_{n\to\infty} \underset{D}{K_n}(t-y) = \delta(t-y),$$

however, (4) simulates a Dirac delta only until certain approximation, then the convergence:

$$\lim_{n\to\infty} f_n(y) = f(y) \tag{6}$$

has to be restricted to a definite class of functions $f(y)$ which are conveniently smooth to counteract the insufficient focusing power of $\underset{D}{K_n}(t-y)$; the corresponding restrictions on $f(y)$ are the known Dirichlet conditions [1–3] for infinite convergent Fourier series.

From (4) we see that $\underset{D}{K_n}(\theta)$ is an even function, then here we consider it for $\theta \in [\,0,\pi\,]$:

$$\underset{D}{K_n}(\theta) = \frac{1}{2\pi}\,\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}, \tag{7}$$

thus

$$\underset{D}{K_0}(\theta) = \frac{1}{2\pi},\quad \underset{D}{K_1}(\theta) = \frac{1}{2\pi}(1+2\cos\theta),\quad \underset{D}{K_2}(\theta) = \frac{1}{2\pi}(-1+2\cos\theta+4\cos^2\theta), \tag{8}$$

$$\underset{D}{K_3}(\theta) = \frac{1}{2\pi}(-1-4\cos\theta+4\cos^2\theta+8\cos^3\theta),\quad \text{etc.}$$

then it is natural to introduce the polynomials:

$$W_n(x) = W_n(\cos\theta) = 2\pi\,\underset{D}{K_n}(\theta),\quad x \in [-1,1] \tag{9}$$

which were named "fourth-kind Chebyshev polynomials" by Gautschi [4,5]. Therefore, see Fig. 1:

$$W_0(x) = 1,\quad W_1(x) = 2x+1,\quad W_2(x) = 4x^2+2x-1, \tag{10}$$

$$W_3(x) = 8x^2+4x^2-4x-1,\quad W_4(x) = 16x^4+8x^3-12x^2-4x+1,\quad \text{etc.}$$

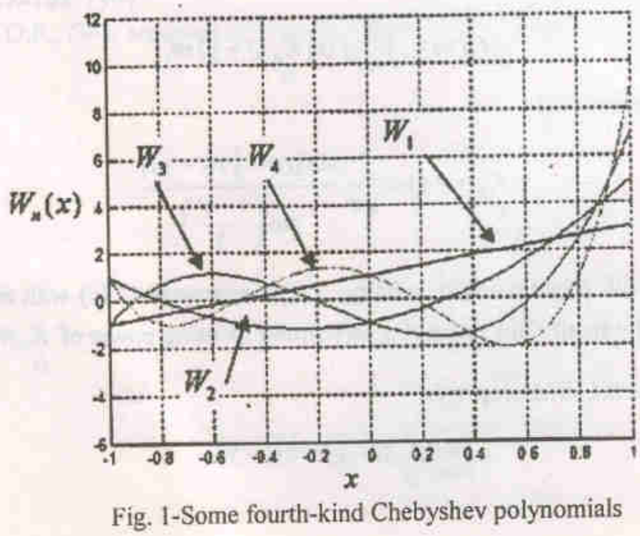

Fig. 1-Some fourth-kind Chebyshev polynomials

In the next Section we exhibit a set of associated polynomials to Fejér kernel [1-3]

Chebyshev- Fejér polynomials

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable.

Then, in the Fejér approach we construct the sequence:

$$(11) \quad g_1(y) = f_0(y),\ g_2(y) = \tfrac{1}{2}[(f_0(y) + f_1(y)],\ g_3(y) = \tfrac{1}{3}[(f_0(y) + f_1(y) + f_2(y)], \ldots, \quad g_n(y) = \tfrac{1}{n}[(f_0(y) + f_1(y) + \ldots + f_{n-1}(y)],$$

accepting the expressions (1) and (2), therefore,

$$(12) \qquad g_n(y) = \int_{-\pi}^{\pi} f(t) \underset{F}{K_n}(t-y)\, dt,$$

thus we see that Fejér results come about by the fact that his method is related with the following kernel [1–3]:

$$(13) \qquad \underset{F}{K_n}(t-y) = \frac{1}{2\pi n} \frac{\sin^2[\frac{n}{2}(t-y)]}{\sin^2 \frac{t-y}{2}},$$

which possesses a strong focusing power, that is, it satisfies (5), then a $f(y)$ absolutely integrable in $[-\pi, \pi]$ guarantees the convergence of $g_n(y)$ towards $f(y)$.

Now we consider the Fejér kernel:

$$(14) \qquad \underset{F}{K_n}(\theta) = \frac{1}{2\pi n} \frac{\sin^2\left(n\frac{\theta}{2}\right)}{\sin^2 \frac{\theta}{2}}, \quad \theta \in [\,0,\pi\,]$$

that is

$$(15) \qquad \underset{F}{K_0}(\theta) = 0,\ \underset{F}{K_1}(\theta) = \frac{1}{2\pi},\ \underset{F}{K_2}(\theta) = \frac{1}{2\pi}(1+\cos\theta), \quad \underset{F}{K_3}(\theta) = \frac{1}{6\pi}(1 + 4\cos\theta + 4\cos^2\theta), \text{ etc.}$$

then it is natural the introduction of the functions:

$$(16) \qquad \bar{W}_n(x) = \bar{W}_n(\cos\theta) = \frac{2\pi}{n+1} \underset{F}{K_{n+1}}(\theta), \quad x \in [\,-1,1\,]$$

that we name "fifth-kind Chebyshev polynomials", which are not explicitly in the literature.

Therefore:

$$\tilde{W}_0(x)=1,\ \tilde{W}_1(x)=\frac{1}{2}(x+1),\ \tilde{W}_2(x)=\frac{1}{9}(4x^2+4x+1),$$

(17)

$$\tilde{W}_3(x)=\frac{1}{2}(x^3+x^2),\ \tilde{W}_4(x)=\frac{1}{25}(16x^4+16x^3-4x^2-4x+1),\text{ etc.}$$

Thus $\tilde{W}_n(1)=1$, see the following Figure:

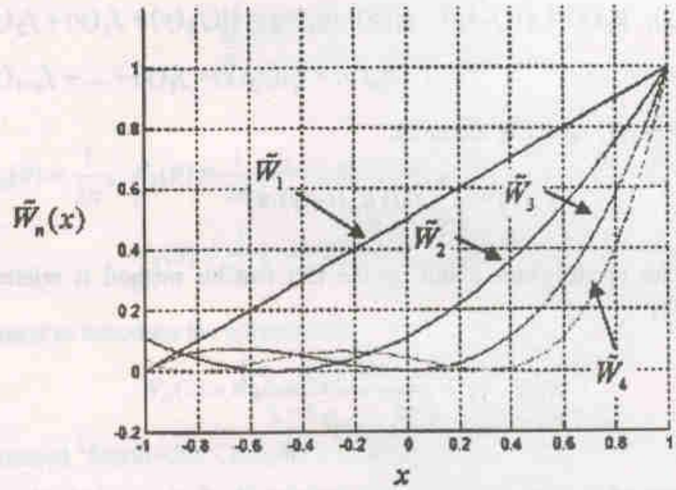

Fig. 2- Some fifth-kind Chebyshev polynomials

which are solutions of the non-homogeneous differential equation:

$$(1-x)[(1-x^2)\tilde{W}_n''-(3x+2)\tilde{W}_n'+(n+1)^2\tilde{W}_n]+x\tilde{W}_n=1. \tag{18}$$

In other paper we will study topics as recurrence, Rodrigues formula, interpolation properties, orthogonality, generating function, etc., for fifth-kind Chebyshev polynomials introduced in this work.

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