The Nepali Math. Sc. Report
Vol. 28. No. 1 & 2. 2008 (Dec.)

Asymptotic behavior of the solution of a Cauchy problem for a Sobolev type system

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Abstract: Asymptotic behavior of the solution of a Cauchy problem for a mathematical model of rotating inviscid compressible fluid is studied for a large time.

Key words: Sobolev type system, Cauchy problem, Asymptotic behavior, Bessel's function, Bessel's equation, Spherical co-ordinates, Taylor's series

1. Introduction

Our everyday life is full of examples of fluid motion, such as stirring a cup of tea, flows in rivers, waves in oceans, hurricanes and many others. The equations that describe the most fundamental behaviour of a fluid were first derived by Euler in 1755. The initial boundary value problems for the Euler equations are surprisingly difficult. Even the basic questions of existence and uniqueness of solutions in three dimensions still remain open.

The idea of wide application of mathematical models of rotating fluids to the study of atmospheric processes stems from A. Friedman, who in the early 20[th] century contributed a series of fundamental works in this direction. In connection with the investigation of a rotating body filled up with fluid, S.L. Šobolev initiated the study of the following system:

$$\frac{\partial \vec{v}}{\partial t} - [\vec{v}, \vec{\omega}] + \nabla p = \vec{F}(x,t); div\vec{v} = 0, x \in \Omega \subset \Re^3, t \geq 0.$$

Now-a-days, this system is known as Sobolev system. Different types of initial and initial-boundary value problems associated with Sobolev system have been studied by various mathematicians in the last five decades.

In a Cauchy problem, i.e. large volumes of rotating fluids (atmosphere, ocean etc.), even a numerical approach to the problem requires studying the solution's behavior as time $t \to \infty$, obtaining at least the leading term of the solution's asymptotic expansion with respect to the small parameter $1/t$ for large time t. In present work, the Cauchy problem for a Sobolev type system describing the motion of a rotating inviscid compressible fluid is taken under consideration and the asymptotic behavior of its solution is studied, as $t \to \infty$. Systems of such a kind are often employed in modeling the dynamics of atmosphere, ocean and environment protection. One of such hydrodynamic models was introduced by G. I. Marchhuk [6].

The following system of equations

(1) $$\frac{\partial \vec{v}}{\partial t} - [\vec{v}, \vec{\omega}] + \nabla p = \vec{F}(x,t); \alpha^2 \frac{\partial p}{\partial t} + div\vec{v} = \Psi(x,t)$$

is considered on the domain $D = \{x \in \Re^3, t \geq 0\}$, where

$\vec{v}(x,t) = (v_1, v_2, v_3)$ - velocity vector field of the fluid,
$\vec{\omega} = (0,0,1)$ - angular velocity of space,
$p(x,t) = (p_1, p_2, p_3)$ - hydrodynamic pressure,
$\vec{F}(x,t) = (F_1, F_2, F_3)$ - mass density of external forces,
$\alpha^2 = const$ - coefficient of compression, and
$[\vec{v}, \vec{\omega}]$ - the vector product of $\vec{v}$ and $\vec{\omega}$.

System (1) belongs to the hyperbolic type according to the classification of partial differential equations. S. L. Sobolev [1,2] was the first to study this system for the particular case $\alpha = 0$. The Cauchy problem and corresponding boundary value problems in certain domains for system (1) were studied by V. N. Maslennikova [3, 4, 5]. Various types of applications of the solutions of such types of system have been discussed by G. I. Marchhuk in his famous book [6].

2. Cauchy Problem and Explicit Solution

Let the initial conditions for the system (1) be

(2) $$\vec{v}(x,0) = \vec{v}^0(x), p(x,0) = p^0(x).$$

The explicit solution of the Cauchy problem (1),(2) was constructed in explicit form by V.N. Maslennikova. Assuming that the functions involved in the initial conditions (2) and the right hand side member of system (1) are sufficiently smooth, the solution of the system (1), (2) was obtained in the following explicit form [4]:

$$(3)\quad \vec{v}(x,t)=\frac{1}{4\pi}\iint\limits_{r=t/\alpha}\left\{-\frac{1}{r}\frac{\partial \vec{v}^0}{\partial n}+\left(\frac{1}{r^2}-\frac{\alpha^2\rho^2}{2r^2}\right)\vec{v}^0+\frac{\alpha}{r}\left([\vec{v}^0,\vec{\omega}]-\nabla p^0\right)\right\}ds_y$$
$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[G(x-y,t)\frac{\partial\vec{\Phi}^0(y)}{\partial y_3}+\sum_{k=1}^{3}\frac{\partial^k G(x-y,t)}{\partial t^k}\vec{\Phi}_k(y)\right]dy$$
$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\int\limits_0^{t-\alpha r}G(x-y,t-\tau)\vec{f}(y,\tau)d\tau\right]dy$$

$$(4)\quad p(x,t)=\frac{1}{4\pi}\iint\limits_{r\le t/\alpha}\left\{-\frac{1}{r}\frac{\partial p^0(y)}{\partial n}-\frac{1}{\alpha r}div\vec{v}^0(y)+\left(\frac{1}{r^2}-\frac{\alpha^2\rho^2}{2r^2}\right)p^0(y)\right\}ds_y$$
$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[G(x-y,t)\frac{\partial v_3^0}{\partial y_3}+\frac{\partial G}{\partial t}\left(\frac{\partial v_1^0}{\partial y_2}-\frac{\partial v_2^0}{\partial y_1}+\alpha^2p^0\right)+\frac{\partial^2 G}{\partial t^2}div\vec{v}^0+\frac{\partial^3 G}{\partial t^3}\alpha^2p^0\right]dy$$
$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\int\limits_0^{t-\alpha r}G(x-y,t-\tau)f_4(y,\tau)d\tau\right]dy.$$

The vector functions $\vec{\Phi}_k(y)$ in (3) are expressed in terms of initially given functions as follows:

$$\vec{\Phi}_0(y)=-rot\vec{v}^0+\alpha^2p^0\vec{\omega};$$
$$\vec{\Phi}_1(y)=-\Delta\vec{v}^0+\nabla div\vec{v}^0-\alpha^2[\nabla p^0,\vec{\omega}]+\alpha^2v_3^0\vec{\omega};$$
$$\vec{\Phi}_2(y)=\alpha^2(\nabla p^0-[\vec{v}^0,\vec{\omega}]);$$
$$\vec{\Phi}_3(y)=\alpha^2\vec{v}^0.$$

The kernel G is given by $G(x-y,t)=\frac{1}{\rho}\int\limits_0^{\rho\sqrt{(t^2-\alpha^2r^2)}/r}\frac{\eta}{\sqrt{\eta^2+\alpha^2\rho^2}}J_0(\eta)\,d\eta$, where

$\rho^2=\sum\limits_{i=1}^{2}(x_i-y_i)^2$; $r^2=\rho^2+(x_3-y_3)^2$ and J_0 is the Bessel's function of order 0.

The functions $\vec{f}$ and f_4 are expressed in terms of $\vec{F}$ and $\dot{\psi}$ as follows:

$$\vec{f}(y,\tau)=-rot\frac{\partial\vec{F}}{\partial y_3}+\vec{\omega}\frac{\partial\Psi}{\partial y_3}+\Delta\frac{\partial\vec{F}}{\partial\tau}-\nabla div\,(\frac{\partial\vec{F}}{\partial\tau})+$$
$$+[\nabla(\frac{\partial\Psi}{\partial\tau}),\vec{\omega}]-\alpha^2\vec{\omega}\frac{\partial F_3}{\partial\tau}+\nabla\frac{\partial^2\Psi}{\partial\tau^2}-\alpha^2\,[\frac{\partial^2\vec{F}}{\partial\tau^2},\vec{\omega}]-\alpha^2\frac{\partial^3\vec{F}}{\partial\tau^3};$$

$$f_4(y,\tau)=\frac{\partial F_3}{\partial y_3}+\frac{\partial^2F_2}{\partial y_1\partial\tau}-\frac{\partial^2F_1}{\partial y_2\partial\tau}-\frac{\partial\psi}{\partial\tau}+div\frac{\partial^2\vec{F}}{\partial\tau^2}-\frac{\partial^3\psi}{\partial\tau^3}.$$

It was proved by using the energy estimates [5] that the solution of the Cauchy problem (1), (2) is unique in $L_2(Q)$, where $Q = \{(x,t) : x \in \Re^3, 0 \le t \le T\}$.

3. Asymptotic Behavior of the Solution

In present work, the Cauchy problem (1), (2) for the corresponding homogeneous system is taken under consideration and the asymptotic behavior of its solution is determined, as $t \to \infty$. It should be noted that the solution of the system automatically belongs to the space L_2. The representation (3), (4) permits us to obtain a series of interesting properties of solution of the Cauchy problem, mainly, the behavior of the Cauchy problem solution as $t \to \infty$.

Now, let us consider the solution (3) for the homogeneous system (1), taking $\vec{f}, f_4 \equiv 0$. The change of variable: $x - y = \xi$ produces $dy = -d\xi$ and then we integrate the first member in the second integral of (3) with respect to ξ_3 taking into account that $G = 0$ on the surface of the cone. Then, using the Bessel's equation $J_0(\eta) = -J_0''(\eta) - J_0'(\eta)/\eta$, the formula for the solution $\vec{v}(x,t)$ of our Cauchy problem corresponding to the homogeneous system can be explicitly written in the following form:

$$
\begin{aligned}
(5)\quad \vec{v}(x,t) = {} & \frac{1}{4\pi} \iint\limits_{r=t/\alpha} \left\{ -\frac{1}{r}\frac{\partial \vec{v}^0}{\partial n} + \left(\frac{1}{r^2} - \frac{\alpha^2\rho^2}{2r^2} \right) \vec{v}^0 + \frac{\alpha}{r}\left([\vec{v}^0, \vec{\omega}] - \nabla p^0 \right) \right\} ds_y \\
& + \frac{1}{4\pi} \iiint\limits_{r \le t/\alpha} \left\{ \left(-\frac{t\xi_3}{r^3} J_0' - \frac{t\xi_3}{r^2 \xi \sqrt{t^2 - \alpha^2 r^2}} J_0' \right) \vec{\Phi}_0(x+\xi) \right. \\
& + \left(-\frac{1}{r} J_0'' - \frac{1}{\rho\sqrt{t^2 - \alpha^2 r^2}} J_0' \right) \vec{\Phi}_0(x+\xi) + \frac{\rho t}{r^2 \sqrt{t^2 - \alpha^2 r^2}} J_0' \vec{\Phi}_2(x+\xi) \\
& \left. + \left[\frac{\rho^2 t^2}{r^3 (t^2 - \alpha^2 r^2)} J_0'' - \frac{\alpha^2 \rho}{(t^2 - \alpha^2 r^2)^{3/2}} J_0' \right] \vec{\Phi}_3(x+\xi) \right\} d\xi .
\end{aligned}
$$

In (5), the omitted arguments of the Bessel's functions are each equal to $\rho\sqrt{t^2 - \alpha^2 r^2}/r$ and also $r^2 = \rho^2 + \xi_3^2, \rho^2 = \xi_1^2 + \xi_2^2$. The formula for $p(x,t)$ will have similar form as that of $\vec{v}(x,t)$ in (5). Now, we state and prove a theorem on asymptotic behavior of the solution (5), which is the main result of present work.

Basic Theorem

If the initial functions involved in (2) belong to the space C_0^∞, then the solutions $\vec{v}(x,t)$ and $p(x,t)$ of the Cauchy problem for the corresponding homogeneous system (1) equipped with the initial conditions (2) decrease like $C(x)/t$ as $t \to \infty$, where $C(x)$ is a bounded function.

Proof of the Theorem

If the time t is sufficiently large and the given initial functions are finite, then the first integral in (5) will be zero. Let's take, into consideration, the first member in the second integral of formula (5).

We have:

$$-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\frac{t\xi_3}{r^3}J_0''\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\vec{\Phi}_0(x+\xi)d\xi$$

$$=-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\frac{\partial}{\partial\xi_3}\left(\frac{\sqrt{t^2-\alpha^2r^2}}{\rho t}\vec{\Phi}_0(x+\xi)\right)d\xi.$$

Then,

$$\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}G_0(\xi,t)\vec{\Phi}_0(x+\xi)d\xi$$

$$=-\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\frac{\partial\vec{\Phi}_0}{\partial\xi_3}\frac{\sqrt{t^2-\alpha^2r^2}}{\rho t}J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\right.$$

$$\left.-\frac{\xi_3\sqrt{t^2-\alpha^2r^2}}{\rho t r^2}\vec{\Phi}_0J_0'\left(\frac{\rho\sqrt{t^2-\alpha^2r^2}}{r}\right)\right]d\xi$$

$=Q_1+Q_2$. (Suppose).

Let us take Q_2 under consideration. Changing to spherical system of co-ordinates, we get:

$$Q_2=-\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha}dr\int_0^{\pi}\vec{\Phi}_0dJ_0$$

$$=\frac{1}{2t}\int_0^{t/\alpha}[\vec{\Phi}_0(x_1,x_2,x_3+r)-\vec{\Phi}_0(x_1,x_2,x_3-r)]dr+$$

$$+\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha}dr\int_0^{\pi}J_0\left(\sin\theta\sqrt{t^2-\alpha^2r^2}\right)\frac{\partial\vec{\Phi}_0}{\partial\theta}d\theta.$$

From this expression for Q_2, it follows that $Q_2 = C(x)/t$, where $C(x)$ is uniformly bounded as $|x| \to \infty$. Now, we take Q_1. Let the smoothly finite function $\frac{\partial \vec{\Phi}_0}{\partial \xi_3}$ be denoted by $\vec{\Phi}_{03}$ and we decompose it into Taylor's series in a neighborhood of $\xi_3 = 0$ with the remainder term in an integral form. To simplify the task, the first two arguments $x_1 + \xi_1$ and $x_2 + \xi_2$ of the function $\vec{\Phi}_{03}$ are omitted. We have:

$$\vec{\Phi}_{03}(x_3 + \xi_3) - \vec{\Phi}_{03}(x_3) = \left(\frac{\partial \vec{\Phi}_{03}}{\partial \xi_3}\right)_{\xi_3 = 0} \xi_3 + \int\limits_{x_3}^{x_3+\xi_3} (x_3 + \xi_3 - \eta) \frac{\partial^2 \vec{\Phi}_{03}(\eta)}{\partial \eta^2} d\eta.$$

So, Q_1 can be expressed in the following form:

$$Q_1 = -\frac{1}{4\pi t} \iiint\limits_{r \le t/\alpha} \frac{\sqrt{t^2 - \alpha^2 r^2}}{\rho} J_0'\left(\frac{\rho\sqrt{t^2 - \alpha^2 r^2}}{r}\right) \left[\vec{\Phi}_{03}(x_3) + \left(\frac{\partial \vec{\Phi}_{03}}{\partial \xi_3}\right)_{\xi_3 = 0} \xi_3 + \right.$$
$$\left. + \int\limits_{x_3}^{x_3+\xi_3} (x_3 + \xi_3 - \eta) \frac{\partial^2 \vec{\Phi}_{03}(\eta)}{\partial \eta^2} d\eta \right] d\xi = \sum_{i=1}^{3} Q_{1i} \quad \text{(Suppose)}.$$

First, we consider the term Q_{11} for investigation.

In spherical co-ordinates, after making the substitution $\sin\theta = \gamma, d\theta = \frac{d\gamma}{\sqrt{1-\gamma^2}}$, we get:

$$Q_{11} = -\frac{1}{2\pi t} \int\limits_0^{2\pi} d\varphi \int\limits_0^{t/\alpha} r dr \int\limits_0^1 \frac{\sqrt{t^2 - \alpha^2 r^2} J_0'(\gamma\sqrt{t^2 - \alpha^2 r^2})}{\sqrt{1-\gamma^2}} \vec{\Phi}_{03}(x_1 + r\gamma\cos\varphi, x_2 + r\gamma\sin\varphi, x_3) d\gamma.$$

Since $\vec{\Phi}_{03}$ depends on γ, its Taylor series expansion in the neighborhood of $\gamma = 1$ gives:

$$\vec{\Phi}_{03}(\gamma) - \vec{\Phi}_{03}(1) = \left(\frac{\partial \vec{\Phi}_{03}}{\partial \gamma}\right)_{\gamma=1} (\gamma - 1) + o((\gamma - 1)^2).$$

So, we can use the method of integration by parts in the improper integral involved in Q_{11} above w.r.t. the variable γ and get rid of the polynomial $\sqrt{t^2 - \alpha^2\gamma^2}$ in the numerator. The remaining integral with $\vec{\Phi}_{03}(1)$ can be obtained in explicit form. In fact,

$$Q_{111} = -\frac{1}{2\pi t} \int\limits_0^{2\pi} d\varphi \int\limits_0^{t/\alpha} r dr \int\limits_0^1 \frac{\sqrt{t^2 - \alpha^2 r^2} J_0'\left(\gamma\sqrt{t^2 - \alpha^2 r^2}\right)}{\sqrt{1-\gamma^2}} \vec{\Phi}_{03}(x_1 + r\cos\varphi, x_2 + r\sin\varphi, x_3) d\gamma.$$

But,

$$(2)\qquad \frac{1}{t}\int_0^1 \sqrt{t^2-\alpha^2r^2}\,\frac{J_0'(\gamma\sqrt{t^2-\alpha^2r^2}}{\sqrt{1-\gamma^2}}\,d\gamma=\frac{\cos\sqrt{t^2-\alpha^2r^2}-1}{t}$$

So, we have:

$$Q_{111}=\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/\alpha}r\left[1-\cos\sqrt{t^2-\alpha^2r^2}\,\right]\vec{\Phi}_{03}(x_1+r\cos\varphi,x_2+r\sin\varphi,x_3)\,dr.$$

In this way, we have obtained $Q_{11}=C(x)/t$. Note that for the case when $\alpha=0$ [7], it takes the form $Q_{11}=\dfrac{1-\cos t}{t}C(x)$, because of the fact that there is no delay (lag) of the argument.

Transferring to spherical co-ordinates and then integrating by parts with respect to the variable θ in the integrals Q_{12} and Q_{13}, we can see that these integrals can be also represented in the form $C(x)/t$, where in every case, $C(x)$ will be uniformly bounded with respect to $x\in\Re^3$. Conversion of the terms with potentials G_1-G_3 in similar way completes the proof of the theorem.

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