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Bottleneck product rate variation problem with absolute-deviation objective

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Abstract: The bottleneck product rate variation problem with absolute-deviation objective is pseudo-polynomially solvable. There always exists an optimal sequence with the property that the deviation for every product is no more than one unit. Only the standard instance has optimal value less than $\frac{1}{2}$ if and only if the demands are successive powers of two. In this paper, we establish that there exists no feasible solution for any instance with the deviation less than $\frac{1}{3}$.

Keywords: non-linear integer programming, mixed-model just-in-time production, balanced word.

1. Introduction

Mixed-model just-in-time production system aims to obtain a sequence of a number of different products with multiple copies that minimizes deviation throughout the time, between the actual and the desired production. Such a sequence maintains the final assembly line keeping the rate of usage of parts as constant as possible and affects the entire supply chain as all other levels are also inherently fixed due to the pull nature.

Minimization of the maximum variation in the rate at which different products are produced on the line is known as the bottleneck product rate variation problem (PRVP) [5]. The problem is formulated as a non-linear integer programming [7, 8]. The problem with absolute-deviation objective has been extensively studied in a great number of papers, for instance see [3].

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The problem is reducible to the release date/due date decision problem, which can be solved to optimality with a pseudo-polynomial algorithm. An optimal sequence always exists when the deviation for every product is never more than one unit [9]. It is important to observe instances that have as small optimal values as possible. It has been established that the problem has optimal absolute deviation less than a half if and only if the demands are successive powers of two [1]. In this paper, we establish that there exists no instance that has a feasible solution with the value less than $\frac{1}{3}$. The general version of the problem is *Co-NP* and is still open whether it is *Co-NP*-complete or polynomially solvable but is polynomially solvable when the number of products is fixed [1].

The plan of the paper is as follows. Section 2 reviews the mathematical model. In Section 3 perfect matching method and the bounds have been studied. Section 4 explains the bisection search and the last section concludes the paper.

2. Mathematical Model

Given $d_i \in N$ demand for a product i, $i = 1,\dots,n$, N being the set of positive integers, with total demand $D = \sum_{i=1}^{n} d_i$ and demand ratio $r_i = \frac{d_i}{D}$, let the time horizon be partitioned into D equal units and each product is produced in a unit time. There will be k complete units of various products during the first k, $k = 1,\dots,D$, time units. Let x_{ik} be the quantity of product i produced during the time units 1 through k. Consider f_i, $i = 1,\dots,n$ unimodal symmetric convex function with minimum 0 at 0.

The mathematical model of the bottleneck PRVP [7, 8] is

(1) $\min \max f_i(x_{ik} - kr_i)$

subject to

(1.1) $\sum_{i=1}^{n} x_{ik} = k \qquad k = 1,\dots,D$

(1.2) $x_{i(k-1)} \le x_{ik} \qquad i = 1,\dots,n;\ k = 2,\dots,D$

(1.3) $x_{iD} = d_i;\ x_{i0} = 0 \quad i = 1,\dots,n$

(1.4) $x_{ik} \ge 0$, integer

Constraint (1.1) shows the cumulative production during the time units 1 through k. Constraint (1.2) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (1.3) guarantees that the demands for each product are met exactly. Constraint (1.2) and (1.4) ensure that exactly one unit of a product is scheduled during one time unit. In this paper, we consider most studied absolute-deviation objective function $f_i(x_{ik} - kr_i) = |x_{ik} - kr_i|$.

3. Perfect Matching Method and the Bounds

A pseudo-polynomial algorithm, order-preserving perfect matching together with bisection search yields an optimal solution to the problem. The problem has been shown to be Co-NP but remains open whether it is *Co-NP-Complete* or polynomially solvable [1]. As the input size is $O(n \log D)$ and there are nD variables with $O(nD)$ constraints, existence of a polynomial time algorithm seems unlikely.

The problem is reduced to an order-preserving perfect matching [9]. The problem is constructed in a V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ with $V_1 = \{1, \dots, D\}$; $V_2 = \{(i,j) \mid i = 1, \dots, n; j = 1, \dots, d_i\}$; and $E = \{(k,(i,j)) \mid k \in [E(i,j),\ L(i,j)]\}$, where $E(i,j)$ and $L(i,j)$ are the earliest and the latest starting time, respectively, for (i,j), the j^{th} copy of product i.

For a bound (target value) B, $E(i,j)$ and $L(i,j)$ can be determined by the integral adjustment of the points where the bound B and the curves $|j - kr_i|$, $i = 1, \dots, n$; $j = 0, \dots, d_i$ intersect.

Lemma 1: [1] *For a given bound B, the earliest and the latest starting times are the unique integers* $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$ *and* $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$, *respectively.*

Proof: If (i,j) is produced in the time unit k, $|x_{ik} - kr_i| = |j - kr_i|$, $i = 1, \dots, n$; $j = 0, \dots, d_i$; $k = 1, \dots, D$. For $j = 0$, $x_{ik} = 0$. $E(i,j)$ must satisfy the inequalities $|j - (E(i,j) - 1)r_i| > B$ and $|j - E(i,j)r_i| \le B$. This implies $\frac{j-B}{r_i} \le E(i,j) < \frac{j-B}{r_i} + 1$.

Therefore, $E(i,j) = \left\lceil \frac{j-B}{r_i} \right\rceil$

Similarly, $E(i,j)$ must satisfy the inequalities $|(L(i,j)-1)r_i - (j-1)| \le B$ and $|L(i,j)r_i - (j-1)| > B$. This implies $\frac{j-1+B}{r_i} < L(i,j) \le \frac{j-1+B}{r_i} + 1$.

Therefore, $L(i,j) = \left\lfloor \frac{j-1+B}{r_i} + 1 \right\rfloor$

Finally, $E(i,j) = 1$ and $L(i,j) = D$ hold if $|j - r_i| \le B$ and $|d_i - r_i - j + 1| \le B$, respectively. □

$E(i,j)$ and $L(i,j)$ can be calculated in $O(D)$ time [9].

A modified version of Glover's earliest due date (EDD) rule finds a perfect matching when applied in the V_1-convex bipartite graph with $B < 1$. The algorithm matches each ascending $k \in V_1$ to the unmatched (i, j) with the smallest $L(i, j)$ [9].

The necessary and sufficient condition for the existence of a perfect matching is the following.

Theorem 1: [1] *For $B < 1$, the graph $G = (V_1 \cup V_2, E)$ formed by the problem has a perfect matching if and only if* $\sum_{i=1}^{n}(\lfloor k_2 r_i + B\rfloor - \lceil (k_1 - 1)r_i - B\rceil) \geq k_2 - k_1 + 1$ *and*
$\sum_{i=1}^{n}(\lceil k_2 r_i - B\rceil - \lfloor (k_1 - 1)r_i - B\rfloor) \leq k_2 - k_1 + 1$ *for all* $k_1,\ k_2 \in V_1$, $k_1 \leq k_2$ *and*
$[E(i,j), L(i,j)] \cap [k_1, k_2] \neq \phi$

Proof: Let $K = [k_1, k_2] \subset V_1$. Then $(i, j) \in N(K)$, where

$N(K) = \{(i,j) : (i,j) \in V_2, \exists k \in K, (k,(i,j)) \in E\}$, the neighborhood of an interval K in V_1.

$\Leftrightarrow [E(i,j), L(i,j)] \cap [k_1, k_2] \neq \phi$

$\Leftrightarrow E(i,j) \leq k_2$ and $L(i,j) \geq k_1$

$\Leftrightarrow \dfrac{j - B}{r_i} \leq k_2$ and $\dfrac{j - 1 + B}{r_i} + 1 \geq k_1$

$\Leftrightarrow \lceil (k_1 - 1)r_i + 1 - B\rceil \leq j \leq \lfloor k_2 r_i + B\rfloor$

Therefore, $\sum_{i=1}^{n}(\lfloor k_2 r_i + B\rfloor - \lceil (k_1 - 1)r_i - B\rceil) \geq k_2 - k_1 + 1$

Let $N(K) = [k_1, k_2] \subset V_1$.

Then $(i, j) \in K \subset V_2$

$\Leftrightarrow [E(i,j), L(i,j)] \subset [k_1, k_2] \subset V_1$

$\Leftrightarrow k_1 \leq E(i,j)$ and $L(i,j) \leq k_2$

$\Leftrightarrow k_1 \leq \dfrac{j - B}{r_i}$ and $\dfrac{j - 1 + B}{r_i} + 1 \leq k_2$

$\Leftrightarrow (k_1 - 1)r_i + B < j < k_2 r_i + 1 - B$

$\Leftrightarrow \lfloor (k_1 - 1)r_i + 1 + B\rfloor \leq j \leq \lfloor k_2 r_i - B\rfloor$

Therefore, $\sum_{i=1}^{n}(\lceil k_2 r_i - B\rceil - \lfloor (k_1 - 1)r_i + B\rfloor) \leq k_2 - k_1 + 1$.

Since $E(i, j)$ and $L(i, j)$ are strictly monotonic and the *EDD* rule assigns the lower numbered copies to earlier time units, the perfect matching is order-preserving. The order-preserving perfect matching gives rise a bijection $(i, j) \rightarrow k$, $(i, j) \in V_2$ and $k \in V_1$,

$i = 1,\ldots,n\,;\, j = 1,\ldots,d_i$ and hence creates a feasible solution to an instance of the problem [9]. The lower and the upper bounds are $1 - r_{\max}$ and $1 - \frac{1}{D}$, respectively.

Theorem 2: [9, 1] *For any instance, the lower and the upper bounds are* $1 - r_{\max}$ *and* $1 - \frac{1}{D}$, *respectively.*

Proof: Given a bound B, a copy of some product i should be produced during $k = 1$. Thus, $\min\,(1 - r_i) \le B$, that is, $1 - r_{\max} \le B$.

Let $B = 1 - \frac{1}{D}$.

Since $\lfloor k_2 r_i + 1 - \frac{1}{D}\rfloor \ge k_2 r_i$ as $\lfloor k_2 r_i + 1 - \frac{1}{D}\rfloor = k_2 r_i$ if $k_2 r_i$ is an integer and $\lfloor k_2 r_i + 1 - \frac{1}{D}\rfloor > k_2 r_i$ if $k_2 r_i$ is not an integer, we have

$$\sum_{i=1}^{n}\left(\lfloor k_2 r_i + B\rfloor - \lceil (k_1 - 1) r_i - B\rceil\right) \ge \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i \ge k_2 - k_1 + 1 .$$

Likewise,

Since $\lceil k_2 r_i - 1 + \frac{1}{D}\rceil \le k_2 r_i$ as $\lceil k_2 r_i - 1 + \frac{1}{D}\rceil = k_2 r_i$ if $k_2 r_i$ is an integer and $\lceil k_2 r_i - 1 + \frac{1}{D}\rceil < k_2 r_i$ if $k_2 r_i$ is not an integer, we have

$$\sum_{i=1}^{n}\left(\lceil k_2 r_i - B\rceil - \lfloor (k_2 - 1) r_i + B\rfloor\right) \le \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i \le k_2 - k_1 + 1 . \qquad \square$$

The lower bound is not always attained, however, the optimal value B^* of the problem often coincides with the lower bound for small size instances [4].

The upper bound is not tight. There exists another upper bound $B^* \le 1 - \dfrac{1}{2(n-1)}$ [10].

Therefore, we can write, $B^* \le 1 - \max\left\{\frac{1}{D}, \frac{1}{2(n-1)}\right\}$ for $n \ge 2$.

It is interesting to investigate those instances with the optimal value that does not exceed $\frac{1}{2}$.

Lemma 2: *An instance has no optimal value less than* $\frac{1}{2}$ *if* $\Delta_i = \dfrac{D}{\gcd(d_i, D)}$, $i = 1, \ldots, n$ *is even.*

Proof: We show that any instance with Δ_i even has lower bound $\frac{1}{2}$. Any feasible solution must satisfy $|[kr_i] - kr_i| \le |x_{ik} - kr_i|$ where $[kr_i]$ is the closest integer to kr_i.

If Δ_i is even, $\Delta_i = 2k$ for some k.

$$|[kr_i] - kr_i| = \left|\left[\frac{\Delta_i}{2}\frac{d_i}{D}\right] - \frac{\Delta_i}{2}\frac{d_i}{D}\right| = \left|\left[\frac{\Delta_i}{2}\frac{\delta_i}{\Delta_i}\right] - \frac{\Delta_i}{2}\frac{\delta_i}{\Delta_i}\right| = \left|\left[\frac{\delta_i}{2}\right] - \frac{\delta_i}{2}\right| = \frac{1}{2} \text{ as}$$

$\gcd(\delta_i, \Delta_i) = 1$, δ_i is odd, $i = 1, \dots, n$. □

It is clear that a standard instance i.e. instance with $\gcd(d_i, D) = 1$, $i = 1, \dots, n$ has lower bound $\frac{1}{2}$ if D is even.

One expects that there may exist instances with $B < \frac{1}{2}$ for Δ_i odd since $|[kr_i] - kr_i| = \frac{\Delta_i - 1}{2\Delta_i}$.

For $n = 2$, infinitely many instances with $B^* < \frac{1}{2}$ exist. A sequence with distances $\left\lfloor \frac{D}{d_1} \right\rfloor$ and $\left\lceil \frac{D}{d_1} \right\rceil$ for product 1 with demand d_1 and $\left\lfloor \frac{D}{d_2} \right\rfloor$ and $\left\lceil \frac{D}{d_2} \right\rceil$ for product 2 with demand d_2 is optimal for two product case [6]. Thus, □

Theorem 3: [6] *For $n = 2$, the optimal value of the problem is less than $\frac{1}{2}$ if and only if one of the demands d_1 or d_2 is odd and the other even.*

For $n > 2$, it has been proven, though appeared as the small deviations conjecture that the standard instance has optimal value $B^* < \frac{1}{2}$ if and only if $d_i = 2^{i-1}$, $i = 1, \dots, n$ and $B^* = \frac{2^{n-1} - 1}{2^n - 1}$ [1].

The sufficient condition of the statement is the following.

Theorem 4: [1] *The instance with $d_i = 2^{i-1}$, $i = 1, \dots, n$, $n \geq 2$ has an optimal sequence with a bound $B^* < \frac{1}{2}$.*

Proof: Consider a standard instance $d_i = 2^{i-1}$, $i = 1, \dots, n$, $n \geq 2$ and a bound $B^* = \frac{2^{n-1} - 1}{2^n - 1}$. Obviously, $B^* < \frac{1}{2}$. Let the copy (i, j) be sequenced at the ideal position $\left\lceil \frac{2j-1}{2r_i} \right\rceil = 2^{n-i}(2j - 1)$, $i = 1, \dots, n$.

The copies do not compete for the position. Let $\frac{2j-1}{2^{n-i}} = \frac{2j'-1}{2^{n-i'}}$ for some positions. Then since both $(2j - 1)$ and $(2j' - 1)$ are odd, neither 2^i divides $(2j - 1)$ nor $2^{i'}$ divides $(2j' - 1)$. This implies $i = i'$ and $j = j'$.

Now we show $2^{n-i}(2j - 1)$ lies in between $E(i, j)$ and $L(i, j)$.

$$\frac{j-B}{r_i}=\frac{j-\dfrac{2^{n-1}-1}{2^n-1}}{\dfrac{2^{i-1}}{2^n-1}}=\frac{(2j-1)(2^n-1)+1}{2^i}=2^{n-i}(2j-1)+\frac{1-(2j-1)}{2^i}\le$$

$$2^{n-i}(2j-1)\le 2^{n-i}(2j-1)+1-\frac{2j}{2^i}=\frac{j-1+\dfrac{2^{n-1}-1}{2^n-1}}{\dfrac{2^i-1}{2^n-1}}+1=\frac{j-1+B}{r_i}+1$$

Since $2^{n-i}(2j-1)$ is an integer, $E(i,j)=\left\lceil\frac{j-B}{r_i}\right\rceil\le 2^{n-i}(2j-1)\le\left\lfloor\frac{j-1+B}{r_i}+1\right\rfloor=L(i,j)$.□

For the necessary case, there exist a geometric proof in [6] and a proof based on balanced word in [2].

The geometric proof exploits a natural symmetry of regular polygons inscribed in a circle of circumference D such that each polygon corresponds to a different product having $d_i, i=1,\ldots,n$ corners for product i at $\left\lceil\frac{2j-1}{2r_i}\right\rceil$ points i.e. the ideal positions on the perimeter of the circle. All the products with demands $d_i=2^{i-1}, i=1,\ldots,n$ are sequenced in the ideal positions [6].

The other proof is based on the concept of balanced word. A δ-balanced word on a finite set $\{1,\ldots,n\}$ is an infinite sequence $s=s_1s_2\ldots$ with $s_i\in\{1,\ldots,n\}$ such that every two subsequences of equal length consist of only those letters whose number of occurrences in each subsequence differ by at most a positive integer δ, (see [11]). Consider a finite word W on $\{1,\ldots,n\}$ of length D with d_i occurrences of a letter i and $r_i=\frac{d_i}{D}$, the rate of letter i with $r_1\le\ldots\le r_n$. W is said to be symmetric if $W=W^R$, a mirror reflection of W. An infinite word W is periodic if $w=WW\ldots$ for some W.

It is shown that a periodic, symmetric and balanced word with $r_1<\ldots<r_n, n>2$ exists if and only if $r_i=\frac{2^{i-1}}{2^n-1}$. This is known as Fraenkel's conjecture for symmetric case. Symmetry and balancedness imply that the rates are all different. The small deviations conjecture is shown to be true as a consequence of the Fraenkel's conjecture for symmetric case using the fact that a solution to the problem with $d_i=2^{i-1},\ i=1,\ldots,n,\ n>2$ is periodic, symmetric and balanced word [2].

We find that only the standard instance with $d_i=2^{i-1}$, $i=1,\ldots,n,\ n>2$ has optimal sequence when $B^*<\frac{1}{2}$. What happens for the instances with gcd $(d_i,D)>1$, $i=1,\ldots,n$ is still unresolved. In this paper, one fact is established that no feasible instance exists when $B<\frac{1}{3}$.

Theorem 5: *There is no instance* $(d_1,...,d_n)$ *with* $n \geq 2$ *that has feasible solution with* $B < \frac{1}{3}$.

Proof: Since tight lower bound is $1-r_{\max}$, $1-r_{\max} \leq B$ and $E(i,j) \leq L(i,j)$, that is, $\frac{j-B}{r_i} \leq \frac{j-1+B}{r_i}+1$ for any feasible sequence.

Thus, $1-r_i \leq 2B$, $\forall i, i=1,...,n$ and $1-r_{\min} \leq 2B$.

This implies $\sum_{i'=1}^{n} r_{i'} \leq 2B,\ r_{i'} \neq r_{\min}$.

Therefore, $r_{\max} \leq \sum_{i'=1}^{n} r_{i'} \leq 2B,\ r_{i'} \neq r_{\min}$.

That is, $1-r_{\max} \geq 1-2B$ which is $1-2B \leq B$.

Hence, $\frac{1}{3} \leq B$.

4. The Bisection Search

A bisection search algorithm to find an optimal sequence for the problem must run in the interval $\left[1-r_{\max}, 1-\frac{1}{D}\right]$. The bisection search is performed using integer instead the rational. For the integral selection to solve the decision problems, the interval is $[D-d_{\max}, D-1]$ that requires $O(\log D)$ time. The overall complexity of the problem is $O(D \log D)$ [9].

5. Conclusion

The bottleneck product rate variation problem with absolute-deviation objective is pseudo-polynomially solvable. The problem is *CO-NP* but remains open for its exact complexity. There always exists an optimal sequence when the deviation for every product is never more than one unit. However, only the standard instance has optimal absolute deviation less than $\frac{1}{2}$ if and only if the demands are successive powers of two and no instance is feasible when the deviation is less than $\frac{1}{3}$. Optimal value for the instance with gcd $(d_i, D) > 1$ is still unresolved.

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