The Nepali Math.Sc. Report
Vol. 19. No. 1 & 2, 2001

Certain Sequence Spaces and Matrix Transformations From $Sl_\infty(p)$ to c and C.

SHAILENDRA KUMAR MISHRA AND KAMAL MANI BARAL

Abstract: Necessary and sufficient conditions have been established for an infinite matrix $A = (a_{nk})$ to transform $S\ell_\infty(p)$ into c and C_s.

1. Introduction

The definition and basic properties of paranormed sequence spaces are given in ([5], [6] and [7]). A paranormed sequence space whose topology is normable is called normed sequence space.

The following sequence spaces will be important in our discussion:

$$\ell_\infty = \{x = \{x_k\} - : \sup_k |x_k| < \infty \}$$
$$c = \{x = \{x_k\} : |x_k| \to 1(k \to \infty), \text{ for some } 1 \in C\}$$
$$c_\infty = \{x = \{x_k\} : |x_k| \to 0(k \to \infty)\}$$
$$C_s = \left\{x = \{x_k\} : \left(\sum_{i=i}^{n} xi\right) \text{ is convergent}\right\} (7,[10])$$

If $\{p_k\}$ is a bounded sequence of strictly positive real numbers, then

$$l(p) = \left\{x = \{x_k\} : \sum_{i=i}^{\infty} |x_k|^{p_k} < \infty\right\}$$
$$l_\infty(p) = \{x = \{x_k\} : \sup_k |x_k|^{pk} < \infty\}$$
$$c(p) = \{x = \{x_k\} : |x_k|^{pk} \to \ell(k \to \infty), \text{ for some } 1 \in C\}$$
$$c_o(p) = \{x = \{x_k\} : |x_k|^{pk} \to o\ (k \to \infty)\}$$

$l(p)$ and $c_o(p)$ are linear metric spaces respectively paranormed by

$$\|x\| = \left[\sum_{k=1}^{\infty} \|x_k\|^{pk}\right]^{1/m} \text{ and } \|x\| = \sup_k |x_k|^{p_k/m}$$

where $M = \max(1, \sup_k p_k)$ $l_\infty(p)$ and $c(p)$ are paranormed by $\|x\| = \sup_k |x_k|^{p_k/m}$ if and only if $p_k > 0$.

For detailed discussions on these spaces we refer ([5], [6], [7] and [9]).
We now define some sequence spaces
Given any $x = \{x_k\}$ we shall write $\Delta x = (x_k - x_{k-1})$, where $x_o = 0$. We define

$$Sl_\infty(p) = \{x = \{x_k\} : \Delta x \in l_\infty(p)\}$$
$$Sp(p) = \{x = \{x_k\} : \Delta x \in c(p)\}$$
$$Sc_o(p) = \{x = \{x_k\} : \Delta x \in c_o(p)\}$$

We write $e = (1,1,1,1,1, \ldots)$, then $S\,l\,\infty\,(e) = Sl_\infty$, $Sc(e) = Sc$ and $Sc_o(e) = Sc_o$

2. Dual Space

If X is a sequence space, we define

$$X^\beta = \left\{ a = \{a_k\} \sum_{k=1}^{\infty} a_k x_k \text{ is convergent foreach } x \in X \right\}$$

We call X^β is the β–(or, generalised kö the – Toeplitz) dual space of X.

Theorem 1(i). *Let $p_k > 0$, for every k. Then*

$$[Sl_\infty(p)]^\beta = \bigcap_{m=1}^{\infty} \{a = \{a_k\} : \left[\sum_{k=1}^{\infty} N^{1/p_m}\right] \text{converges and } \sum_{k=1}^{\infty} N^{1/p_k} |R_k| < \infty, N > 1 \}$$

where $R_k = \sum_{v=k}^{\infty} a_v$ $\left(\text{We assume that } \sum_{m=1}^{k} z_m = 0\ (k > 1)\right)$

Proof: Suppose that $x \in Sl_\infty(p)$. We choose $N > 1$, so that $\sup_k |\Delta x_k|^{p_k} < N$
We write

$$(1) \qquad \sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} R_k \Delta x_k - R_{m+1} \sum_{k=1}^{\infty} \Delta x_k \quad (m = 1,2,3, \ldots)$$

Since $\sum_{k=1}^{\infty} |R_k||\Delta x_k| \le \sum_{k=i}^{\infty} |R_k| N^{1/p_k} < \infty$, it follows that

$\sum_{k=1}^{\infty} R_k \Delta x_k$ is absolutely convergent. By corollary 2 in [2], the convergence of

$\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} N^{1/p_m} \right)$ implies that $\lim_{m \to \infty} R_{m+1} \sum_{m=1}^{k} N^{1/p_m} = 0.$

Hence, it follows from (1) that

$$\sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in Sl_\infty(p)$$

This yields $a \in (Sl_\infty(p))^\beta$

Conversely, suppose that $a \in (Sl_{\infty}(p))^{\beta}$, then by definition, $\sum_{k=1}^{\infty} a_k x_k$ is convergent for each $x \in Sl_{\infty}(p)$.

Since $e = (1,1,1,\ldots\ldots\ldots) \in Sl_{\infty}(p)$ and $x = \left[\sum_{m=1}^{\infty} N^{1/p_m} \right] \in Sl_{\infty}(p)$, then $\sum_{v=1}^{\infty} a_v$ and $\sum_{v=1}^{\infty} a_v \left[\sum_{m=1}^{\infty} N^{1/p_m} \right]$ are respectively convergent. By using corollary 2 in [3], we find that

$$\lim_{m \to \infty} R_{m+1} \sum_{m=1}^{v} N^{1/p_m} = 0$$

Thus, we get from (1) that the series $\sum_{k=1}^{\infty} R_k \Delta x_k$ converges for each $x \in Sl_{\infty}(p)$.

Since $x \in Sl_{\infty}(p)$ if and only if $\Delta x \in Sl_{\infty}(p)$. This implies that $R = \{R_k\} \in (l_{\infty}(p))^{\beta}$. It now follows from a theorem 2 in [4] that

$\sum_{k=i}^{\infty} |R_k| N^{1/p_k}$ convergence for all $N > 1$.

Theorem 1(ii). *Let $P_k > 0$, for every k. Then* $Sc_0(p)^{\beta} = SM_0(p)$, where

$$SM_0(p) = \bigcup_{N>1} \left\{ a = \{a_k\} : \sum_{k=1}^{\infty} a_k \left[\sum_{m=1}^{k} N^{-1/p_m} \right] \text{ converges and } \sum_{k=1}^{\infty} |R_k| N^{-1/p_k} < \infty, N > 1 \right\}$$

Proof: Let $a \in SM_0(p)$ and $x \in Sc_0(p)$. We choose an integer N such that $|\Delta x_k|^{p_k} < N^{-1}$.

We have

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{\infty} R_k \Delta x_k - R_{m+1} \sum_{k=1}^{\infty} \Delta x_k : m = 1, 2, 3, 4 \ldots$$

Since

$$\sum_{k=1}^{\infty} |R_k \Delta x_k| \le \sum_{k=i}^{\infty} |R_k| |\Delta x_k| \le \sum_{k=i}^{\infty} |R_k| N^{1/p_k} < \infty, \text{ it follows that,}$$

$\sum_{k=1}^{\infty} R_k \Delta x_k$ is absolutely convergent. The convergence of

$\sum_{k=1}^{m} a_k \left[\sum_{m=1}^{k} N^{-1/p_m} \right]$ implies that $\lim_{m \to \infty} R_{m+1} \sum_{i=1}^{m} N^{-1/p_i} = 0(1)\ m \to \infty$. Hence $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in Sc_0(p)$. That is, a $a \in Sc_0(p)^{\beta}$

Conversely, let $a \in Sc_0(p)^\beta$. Then, for any $x \in Sc_0(p)$, $\sum_{k=1}^{\infty} a_k x_k$ converges.

Since $x = \left\{ \sum_{m=1}^{k} N^{-1/p_m} \right\}$ by choosing $\in > 1/N, N = 2, 3, \ldots \in Sc_0(p)$ it follows

that $\sum_{k=1}^{\infty} a_k \left(\sum_{m=1}^{k} N^{-1/p_m} \right)$ converges. To show that $\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} < \infty$, $N>1$,

we suppose that, $\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} = \infty$, $N>1$. Then from Theorem 6 in [8], it

follows that $R \notin M_0(p) = [c_0(p)]\beta$. Then there is a sequence $x = \{1/k\}$,

$k \geq 1 \in c_0(p)$ such that $\sum_{k=1}^{\infty} R_k 1/k$ does not converge. Although, if we define

$y = \{y_k\}$ by $y_k = \sum_{n=1}^{k} 1/n$, then $y \in Sc_0(p)$

But

$$\sum_{k=1}^{\infty} a_k y_k = \sum_{k=1}^{\infty} a_k \left\{ \sum_{n=1}^{k} 1/n \right\} = \sum_{k=1}^{\infty} R_k 1/k.$$

Hence, $\sum_{k=1}^{\infty} a_k y_k$ does not converge for $y \in Sc_0(p)$, a contradiction to the fact that

$a \in Sc_0(p)^\beta$. So,

$$\sum_{k=i}^{\infty} |R_k| N^{-1/p_k} < \infty, \quad N>1.$$

This completes the proof.

Matrix Maps

Let X and Y be any two sequence spaces. Let $A = (a_{n,k})$ be an infinite matrix of scalar entries. If $x = \{x_k\} \in X$, then $A_x = (A_n(x))_{n=1}^{\infty} \in Y$

where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n = 1,2,3, \ldots$. We say that A defines a matrix map from X into Y and we write $A \in (X, Y)$. By (X,Y) we mean the class of matrices A such that $A \in (X, Y)$.

The main aim of this section is to characterize the $(Sl_\infty(p), c)$ and $(Sl_\infty(p), Cs)$ We shall first establish the following simple characterization. In order to characterize, we need the following lemma.

Lemma 1. Let X and Y be sequence spaces, and let

$\Delta Y = (y = \{y_k\} : \Delta y = \{y_k - y_{k-1}\}) \in Y, y_0 = 0\}$. Then $A \in (X, Y)$ if and only if $\Delta A = (a_{n,k} - a_{n-1,k}) = (b_{n,k}) = B \in (X, \Delta Y)$.

With lemma 1 and Theorem 1(i,ii) in [4] or Theorem 3 in [4] or Theorem 5b(i) and Theorem 7 in [6], a characterization of the class. $(1(p) ; Sl_\infty)$ or $(l_\infty(p) ; Sl_\infty)$ or $(1(p) ; Sl_\infty(q))$ is immediately follows $q \in l_\infty$.

Theorem 2. *Let* $p_k > 0$, *for every k. Then* $A \in (Sl_\infty(p), c)$ *if and only if*

(i) $R \in (l_\infty(p), c)$

(ii) $A_n\left[\sum_{i=1}^{k} N^{-1/p_i}\right] \in c$ $(n,k = 1, 2, 3, \ldots.)$, for all integers, $N > 1$,

(iii) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ $(k = 1, 2, 3, \ldots\ldots)$

Where,

$$R = (r_{n,k}) = \left[\sum_{v=k}^{\infty} a_{n,v}\right] \quad (n,k = 1, 2, 3, \ldots\ldots)$$

Proof: Let us first prove the sufficiency. Consider any $x \in Sl_\infty(p)$. We choose $N > 1$, so that

$$\sup_k |\Delta x_k|^{p_k} < N.$$

We, write,

$$(2) \qquad \sum_{k=1}^{\infty} a_{n,k}\, x_k = \sum_{k=1}^{m} r_{n,k}\, \Delta x_k - r_{n+1,m} \sum_{k=1}^{m} \Delta x_k \ \ (m = 1, 2, 3, 4 \ldots\ldots)$$

By condition (ii), $\sum_{k=1}^{\infty} a_{n,k}\left[\sum_{i=1}^{k} N^{-1/p_r}\right]$ is convergent for each $n = 1, 2, 3, \ldots.$

Hence, by corollary 2 in [3] it follows that

$$\lim_{m\to\infty} \ r_{n+1},\ m\sum_{i=1}^{m} N^{1/p_i} = 0$$

By condition (i) $R \in (l_\infty(p), c)$, and since $x \in Sl_\infty(p)$ if and only if $\Delta x \in Sl_\infty(p)$.

Hence, by corollary 2 in [3] it follows that $\sum_{k=1}^{\infty} |x_{n,k}|\, N^{1/p_k}$ is uniformly convergent in n and $\lim_{n\to\infty} r_{n,k}$ exists for each $k = 1, 2, 3, \ldots$

Since,

$$\sum_{k=1}^{\infty} |r_{n,k}||\Delta x_k| \le \sum_{k=1}^{\infty} |r_{n,k}|\, N^{1/p_k}$$

Thus, from (2) we find that $\sum_{k=1}^{\infty} a_{n,k} x_k$ is absolutely and uniformly convergent in n.

Finally, we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k}\, x_k = \sum_{k=1}^{\infty} a_{n,k}\, x_k .$$

This proves the sufficiency.

The necessities of (iii) and (ii) are respectively obtained by taking

$x = e = (1,1,1,\ldots\ldots) \in Sl_\infty(p)$ and $x = \left[\sum_{i=1}^{k} N^{-1/p_r} \right] \quad (k = 1,2,3,\ldots\ldots) \in Sl_\infty(p)$.

Now consider the necessity of (I). If it is not true, then there exists $x = \{x_v\} \in l_\infty(p)$ with $\sup_v |x_k|^{p_v} = 1$ such that $\left[\sum r_{n,v} x_v \right]_{n=1}^{\infty} \notin c$. Although if we define a sequence $y = \{y_k\}$ by

$$y_v = \sum_{i=1}^{v} x_i \quad (v = 1,2,3,\ldots\ldots),$$

then $y \in Sl_\infty(p)$ but that,

$$\left[\sum_{v=1}^{\infty} a_{n,v} y_v = \sum_{v=1}^{\infty} r_{n,v} r_v \right] \notin c.$$

This contradicts the fact that $A \in (Sl_\infty(p), c)$ and therefore (I) must hold.

Before characterizing the class $(Sl_\infty(p)$, Cs) we add one more notation. For any $n > 1$, we write

$$t_n(AX) = \sum_{i=1}^{N} A_i(x) = \sum_{k=1}^{\infty} b_{n,k}\, x_k [x \in Sl_\infty(p)],$$

where $$B = (b_{n,k}) = \left[\sum_{i=1}^{k} a_{i,k} \right] (n, k = 1,2,3,\ldots)$$

Theorem 3. *Let $p_k > 0$, for every k. Then $A \in (Sl_\infty(p), Cs)$ if and only if*

(i) $C \in (l_\infty(p), \mathrm{Cs})$,

(ii) $B_n \left[\sum_{i=1}^{k} N^{\frac{1}{p_i}} \right] \in Cs \;\; (n, k = 1,2,3,\ldots) \; N > 1$

(iii) $\lim_{n\to\infty} b_{n,k} = \lim_{n\to\infty} \sum_{i=1}^{n} a_{i,k} = \beta_k \qquad (k = 1,2,3,\ldots)$

where,

$$C = (C_{n,k}) = \left\{ \sum_{i=1}^{n} \left[\sum_{v=k}^{\infty} a_{iv} \right] \right\} \qquad (n, k = 1,2,3,\ldots)$$

This theorem follows immediately from Theorem 2 and Theorem 2 in [8].

We remark that in proving theorem 3, we use $\sum_{k=1}^{m} b_{n,k} x_k = \sum_{k=1}^{m} C_{n,k} \Delta x_k - C_{n,m+1} \sum_{k=1}^{m} \Delta x_k$

$(m = 1,2,3, \ldots\ldots)$ and the convergence of

$$\sum_{k=1}^{\infty} b_{n,k} \left[\sum_{i=1}^{m} N^{\frac{1}{p_i}} \right] \text{ implies that } \lim_{m\to\infty} C_{n,m+1} \sum_{i=1}^{m} N^{\frac{1}{p_i}} = 0$$

Characterization of $(1(p), Sc_0(q)), q \in l_\infty$ follows from Theorem 5(ii) in ([6]) with Lemma 1.

REFERENCES

[1] B Choudhary and S.Nanda, *Functional Analysis with Applications*, Wiley Eastern Limited, 1989.

[2] B. Choudhary and S.K. Mishra, International J. Math. Sci. (18) [1995] No.4. 681–688.

[3] H.Kizmaz, *On certain sequence spaces*, Canadian Math. Bull. 24(2) (1981), 169–176.

[4] C.G.Laascarides and I.J. Maddox, *Matrix transformation between some class of sequences*. Proc. Cambridge, Phil. Soc.68(1970), 99–104.

[5] I.J. Maddox, *Paranormed sequence spaces generated by infinite matrices*, Proc. Cambridge Phil Soc. 64 (1968), 335–340.

[6] I.J. Madox and Michael A.L. Willey, *Continuous operators on Paranormed spaces and Matrix transformations*, Pacific Math. Vol. 53 No. 1 (1974), 217–228.

[7] I.J. Madox, *Elements of functional Analysis*, Cambridge, 1970.

[8] S. Nanda, *Matrix transformations in some sequence spaces*, Indian J.Pure appl. Math. 20(7), 707–710.

[9] S. Simons, *The sequence spaces* $1(p_v)$ *and* $m(p_v)$, Proc. London Math. Soc, 3(15) (1965), 422–436.

[10] M.Stieglitz and H. Tietz, Math. Z. 154(1977), 1–16.

SHAILENDRA KUMAR MISHRA AND KAMAL MANI BARAL
Department of Mathematics
Pulchowk Campus, I.O.E.
Nepal