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Contact CR-warped product submanifolds in locally conformal almost cosymplectic manifolds

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Abstract: B. Y. Chen [2] studied warped products, which are CR-submanifolds in Kaehler manifolds and established sharp inequalities for CR-warped products in Kaehler manifolds. In this article, we establish the inequality for the squared norm of the second fundamental form in terms of warping function for contact CR-warped products isometrically immersed in locally conformal almost cosymplectic manifold.

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1. Introduction

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact manifold equipped with an almost contact structure (ϕ, ξ, η), that is, ϕ is a (1, 1) tensor field, ξ is a vector field and η is a 1-form such that $\phi^2 X = -X + \eta(X)\xi$ and $(\xi) = 1$, then $\phi(\xi) = 0$ and $\eta \, o \, \phi = 0$. The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ defined by

$$J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X - \lambda \xi x, \eta(X) \frac{d}{dt}\right) \tag{1.1}$$

is integrable, where X is tangent to $\bar{M}$, t the coordinate of $\mathbb{R}$ and λ is a smooth function on $\bar{M} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor

$$[\phi, \phi] + 2d\eta \otimes \xi, \tag{1.2}$$

where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ

Let g be a compatible Riemannian metric with (ϕ,ξ,η), that is

(1.3) $$g(\phi X,\phi Y)=g(XY)-\eta(X)\eta(Y).$$

Let Φ be the fundamental 2-form of $\bar{M}$, defined by

(1.4) $$\Phi(X,Y)=g(X,\phi Y)=-g(\phi X,Y)$$

for all $X,Y\in T\bar{M}$..

If Φ and 1-form η are closed, then $\bar{M}$ is said to be almost cosymplectic manifold. A normal almost cosymplectic manifold is cosymplectic. $\bar{M}$ is called a locally conformal almost cosymplectic if there exists a 1-form ω such that

(1.5) $$d\Phi=2\omega\wedge\Phi,\ d\eta=\omega\wedge\eta \ \text{ and } \ d\omega=0.$$

A necessary and sufficient condition for an almost contact structure to be normal locally conformal almost cosymplectic is [6]

(1.6) $$(\bar{\nabla}_X\phi)Y=u(g(\phi X,Y)\xi-\eta(Y)\phi X)$$

where $\bar{\nabla}$ is the Levi-Civita connection of the Riemannian metric g and $\omega=u\eta$.

From (1.6) it follows that

(1.7) $$\bar{\nabla}_X\xi=u\{X-\eta(X)\xi)\}.$$

A plane section π in $T_p\bar{M}$ is called a ϕ-section if it is spanned by X and ϕX, *where* X is the unit tangent vector orthogonal to ξ. The sectional curvature of the ϕ-section is called a ϕ-sectional curvature. A locally conformal almost cosymplectic manifold $\bar{M}$ of dimension ≥ 5 is of point wise constant ϕ-sectional curvature if and only if its curvature tensor $\bar{R}$ is of the form [8]

(1.8) $$\begin{aligned}\bar{R}(X,Y,Z,W)=&\frac{c-3u^2}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\\&+\frac{c+u^2}{4}\{g(X,\phi W)g(Y,\phi Z)-g(X,\phi Z)g(Y,\phi W)-2g(X,\phi Y)g(Z,\phi W)\}\\&-\left(\frac{c+u^2}{4}+u\right)\{g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(Y)\eta(W)+g(Y,Z)\eta(X)\eta(W)\\&-\{g(Y,W)\eta(X)\eta(Z)\},\end{aligned}$$

where u is the function such that $\omega=u\eta$ and $u=\xi u$ and c is the point wise constant ϕ-sectional curvature of $\bar{M}$.

Let M be a n-dimensional submanifold of a manifold $\bar{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by

$$(1.9)\qquad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\bar{\nabla}, \nabla$ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\bar{M}, M$ and normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to the shape operator A by $g\{h(X,Y),N\} = g(A_N X, Y)$. Then the equation of Gauss is given by

$$(1.10)\qquad R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W),\, h(Y,Z)) - g(h(X,Z), h(Y,W)),$$

for any vector fields X, Y, Z and W tangent to M.

Let $\{e_1, \ldots, e_n, \ldots, e_{2m+1}\}$ be an orthonormal basis of the tangent space $T_p\bar{M}$, such that $e_1, e_2, \ldots, e_n$ are tangent to M at p. The mean curvature vector $H(p)$ at $p \in M$ is

$$(1.11)\qquad H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

The submanifold M is totally geodesic in $\bar{M}$ if $h = 0$, and minimal if $H = 0$. We set

$$(1.12)\qquad h_{ij}^{r} = g\big(h(e_i,e_j),e_r\big), \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n}\big(g(h(e_i,e_j),h(e_i,e_j)\big)$$

Let M be a Riemannian manifold of dimension n and $'a'$ a smooth function on M. Now, we recall

(i) ∇_a, the gradient of a is defined by

$$g(\nabla_a, X) = X(a),$$

for all vector field X on M.

(ii) Δ_a, the Laplacian of a is defined by

$$\Delta_a = \sum_{j=1}^{n}\big\{(\nabla_{e_j} e_j)a - e_j e_j(a)\big\} = -\operatorname{div} \nabla_a,$$

where ∇ is the Levi-Civita connection on M and $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame on M.

Consequently, we have

$$\|\nabla_a\|^2 = \sum_{j=1}^{n}\big(e_j(a)\big)^2$$

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds:

(i) A submanifold M tangent to ξ is called an invariant submanifold if $\phi(T_pM) \subset T_pM$ for all $p \in M$, i.e. ϕ-preserves the tangent space of M.

(ii) A submanifold M tangent to ξ is called an anti-invariant submanifold if $\phi(T_pM) \subset T_p^{\perp}M$ for all $p \in M$, where T_pM and $T_p^{\perp}M$ denote tangent and normal space at $p \in M$, respectively.

(iii) A submanifold M tangent to ξ is called a contact CR - submanifold if it admits an invariant distribution D whose orthogonal complimentary distribution $D^{\perp}$ is anti-invariant, that is, $T_pM = D_p \oplus D_p^{\perp}$, with $\phi(D_p) \subset D_p$ and $\phi(D_p^{\perp}) \subset T_p^{\perp}M$, for every $p \in M$.

2. Contact CR-warped product submanifolds:

B.Y. Chen established a sharp relationship between the warping function f of a warped product CR-submanifold $M_1 \times {}_f M_2$ of a Kaehler manifold $\bar{M}$ and the squared norm of the second fundamental form $\|h\|^2$ (see [2]).

We prove a similar inequality for contact CR-warped product submanifolds in locally conformal almost cosymplectic manifold.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of positive dimension n_1 and n_2 respectively and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times {}_f M_2 = (M_1 \times M_2, g)$, where $g = g_1 + f^2 g_2$ (see [3] and [4]).

We recall the following general formulae on a warped product

$$\nabla_U V = \nabla_V U = (U \ln f)V, \tag{2.1}$$

for any vector fields U tangent to M_1 and V tangent to M_2.

In this section, we investigate warped products $M = M_1 \times {}_f M_2$ which are contact CR-submanifolds of a locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c. Such submanifolds are tangent to the structure vector field ξ. We distinguish two cases:

(a) ξ is tangent to M_1.

(b) ξ is tangent to M_2..

In case (a), we consider two sub cases:

(1) M_1 is an anti-invariant submanifold and M_2 is an invariant submanifold of $\bar{M}$.

(2) M_1 is an invariant submanifold and M_2 is an anti-invariant submanifold of $\bar{M}$.

We start with the sub case (1)

Theorem 2.1 *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of point wise constant ϕ-sectional curvature c. Then there do* *not exist warped product submanifold $M = M_1 \times {}_f M_2$ such that M_t is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\bar{M}$.*

Proof : Let $M = M_1 \times {}_f M_2$ be a warped product submanifold of a locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c,such that M_1 is an anti-invariant submanifold tangent to ξ and M_2 an invariant submanifold of $\bar{M}$. From equation (2.1) we have

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{2.2}$$

for any vector fields Z and X tangent to M_1 and M_2 respectively.

In particular, for $z = \xi$, we get $\xi f = 0$. Using (1.7), (1.9) and (2.2), we have

$$u(X - \eta(X)\xi) = \bar{\nabla}_X \xi = \nabla_X \xi = (\xi \ln f)X = 0.$$

Thus M_2 cannot exist.

Now for sub case (2), we have

Theorem 2.2 *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold of point wise constant ϕ-sectional curvature c and $M = M_1 \times {}_f M_2$ an n-dimensional warped product submanifold such that M_1 is a $(2\alpha+1)$-dimensional invariant submanifold tangent to ξ and M_2 a β-dimensional C-totally real submanifold of $\bar{M}(c)$.*

Then

(i) The squared norm of the second fundamental form of M satisfies

$$\|h\|^2 \geq 2\beta\left[\left\|\nabla(\ln f)\right\|^2 - \Delta(\ln f)\right] + \alpha\beta\left(c + u^2 + 4\right) \tag{2.3}$$

where Δ denotes the Laplace operator on M_1.

(ii) The euqlity of (2.3) holds identicallyif M_1 is a totally geodesic submanifold of $\bar{M}(c)$. Hence M_1 is a locally conformal almsot cosymplectic manifold of point wise constant ϕ-sectioanl curvature c.

Proof: Let $M = M_1 \times {}_f M_2$ be a contanct CR-warped product submanifold in locally conformal almost cosymplectic manifold $\bar{M}(c)$ of point wise constant ϕ-sectional curvature c such that dim $M_2 = (2\alpha+1)$ and dim $M_2 = \beta$. Let $\{X_0 = \xi, X_1, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on M such that $X_0, \ldots, X_{2\alpha}$ are tangent to M_1 and $Z_1, \ldots, X_\beta$ tangent to M_2. For any unit vector fields X tangent to M_1 and Z, W tangent to M_2 respectively, we have

$$\text{(2.4)} \qquad g(h(\phi X,Z),\phi Z) = g(\overline{\nabla}_Z \phi X,\phi Z) = g(\phi \overline{\nabla}_Z X,\phi Z)$$
$$= g(\overline{\nabla}_Z X,Z) = g(\overline{\nabla}_Z X,Z) = X \ln f.$$

On the other hand, since the ambient manifold $\overline{M}(c)$ is a locally conformal almost cosymplectic manifold, it is easily seen that

$$\text{(2.5)} \qquad h(\xi,Z) = 0$$

We denote by $h_{\phi D^{\perp}}(X,Z)$ the component of $h(X,Z)$ in $\phi D^{\perp}$. Therefore from (2.4) and (2.5) we have

$$g(h(\phi X,Z),\phi W) = g(A_{\phi W} Z,\phi X) = g(\overline{\nabla}_Z \phi W,\phi X)$$
$$\text{(2.6)} \qquad = g(\overline{\nabla}_Z W,X) = (X \ln f)\, g(Z,W)$$

Putting $X = \phi X, W = \phi W$ in (2.6) we have

$$\text{(2.7)} \qquad g(h(X,Z),W) = \phi X(\ln f)\, g(Z,\phi W) = -\phi X(\ln f)\, g(\phi Z,W).$$

Using (2.7) we have

$$h(X,Z) = -\phi X(\ln f)\phi Z.$$

Therefore for $X \in TM_1$, $Z \in TM_2$,

$$\text{(2.8)} \qquad \|h(X,Z)\|^2 = (\phi X(\ln f))^2\, g(\phi Z,\phi Z) = (\phi X(\ln f))^2\, g(Z,Z)$$
$$= (\phi X(\ln f))^2.$$

Let ν be the normal sub bundle orthogonal to $\phi D^{\perp}$. Obviously, we have

$$T^{\perp}M = \phi D^{\perp} \oplus v, \quad \phi v = v.$$

Let $\{e_i\}_{i=0,\ldots,2\alpha}$ and $\{Z_i\}_{i=1,\ldots,\beta}$ are local orthonormal frame on M_1 and M_2 respectively. On M_1, we consider a ϕ-adapted orthonormal frame namely $\{e_i,\phi e_i,\xi\}_{i=1,\ldots,\alpha}$. We calculate $\|h(X,Z)\|^2$ for $X \in D$, and $Z \in D^{\perp}$. Since, we know that

$$h(X,Z) = h_{\phi D^{\perp}}(X,Z) + h_{v}(X,Z),$$

where $h_{\phi D^{\perp}}(X,Z) \in \phi D^{\perp}$ and $h_v(X,Z) \in v$.

For $X \in TM_1$, $Z \in TM_2$, we have

$$\|h(X,Z)\|^2 = \sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta}\left\{ \| h(e_i,Z_t) \|^2 + \| h(fe_i,Z_t) \|^2 \right\} + \sum_{t=1}^{\beta} \| h_{\phi D^{\perp}}(\xi,Z_t) \|^2$$

Now from (2.8), we have

$$\| h_{\phi D^{\perp}}(e_i,Z_t) \|^2 = (\phi\, e_i(\ln f))^2$$
$$\| h_{\phi D^{\perp}}(\phi\, e_i(\ln f) \|^2 = (\phi^2 e_i(\ln f))^2 = (e_i(\ln f))^2$$

Since

$$\|\nabla_a\|^2=\sum_{i=1}^{2a}(e_i(a))^2.$$

We have

(2.9) $$\|\nabla(\ln f)\|^2=\sum_{i=1}^{2a}(e_i(\ln f))^2+\sum_{i=1}^{2a}(\phi e_i(\ln f))^2$$
$$=\sum_{i=1}^{2a}\sum_{t=1}^{b}\left(\left\|h_{\phi D^\perp}(\phi e_i,\hat{Z_t})\right\|^2+\left\|h_{\phi D^\perp}(e_i,Z_1)\right\|^2\right).$$

Therefore from (2.5) and (2.9), we have

$$\sum_{i=1}^{2a}\sum_{t=1}^{b}\left\|h_{\phi D^\perp}(X_i,Z_t)\right\|^2=\sum_{i=1}^{2a}\sum_{t=1}^{\beta}\left(\left\|h_{\phi D^\perp}(X_i,Z_t)\right\|^2+\left\|h_{\phi D^\perp}(\mathrm{e},Z_t)\right\|^2\right)$$
$$+\sum_{t=1}^{\beta}\left\|h_{\phi D^\perp}(\xi,Z_1)\right\|^2=\sum_{t=0}^{\beta}\left(\|\nabla(\ln f)\|^2\right).$$

From above equation, we have

(2.10) $$\sum_{i=0}^{2a}\sum_{t=0}^{\beta}\left\|h_{\phi D^\perp}(X_i,Z_t)\right\|^2=\sum_{t=0}^{\beta}\|\nabla(\ln f)\|^2=\beta\left(\|\nabla(\ln f)\|^2\right).$$

For any vector field X tangent to M_1 and orthogonal to ξ and Z tangent to M_2, equation (1.8) gives

(2.11) $$\bar{R}(X,\phi X,Z,\phi Z)=\frac{c-3u^2}{4}\{g(\phi X,Z)g(X,\phi Z)-g(X,Z)g(\phi Y,\phi Z)\}+$$
$$+\frac{c+u^2}{4}\{g(X,\phi Z)g(\phi^2X,\phi Z)-g(\phi X,\phi Z)g(\phi X,\phi Z)\}+$$
$$+2g(X,\phi^2X)g(\phi Z,\phi Z)\}$$
$$+\left(\frac{c+u^2}{4}+u\right)\{g(\phi X,\phi Z)\eta(X)\eta(Z)-g(X,\phi Z)\eta(\phi X)\eta(Z)$$
$$+g(X,Z)\eta(\phi X)\eta(\phi Z)-g(\phi X,Z)\eta(X)\eta(\phi Z)\}.$$
$$=2\left(\frac{c+u^2}{4}\right)\{g(X,\phi^2X)g(\phi Z,\phi Z)\}.$$
$$=-\left(\frac{c+u^2}{2}\right)$$

On the other hand, by Codazzi equation, we have

(2.12) $$\bar{R}(X,\phi X,Z,\phi Z)=-g\left(\nabla_X^\perp h(\phi X,Z)-h(\bar{\nabla}_X\phi X,Z)-h(\phi X,\nabla_X Z),\phi Z\right)$$
$$+g\left(\nabla_{\phi X}^\perp h(X,Z)-h(\nabla_{\phi X}X,Z)-h(X,\nabla_{\phi X}Z),\phi Z\right)$$

By using equations (1.6) and (2.1), we get

$$\begin{aligned} g\left(\nabla_X^{\perp} h(X,Z),\phi Z\right) &= Xg\left(h(\phi X,Z),\phi Z\right) - g\left(h(\phi X,Z),\bar{\nabla}_X \phi Z\right) \\ &= Xg(\nabla_Z X,Z) - g\left(h(\phi X,Z),\phi\bar{\nabla}_X Z\right) = X(X\ln f)\, g(Z,Z) \\ &\quad -(X\ln f)\, g\left(h(\phi Y,Z)\phi Z\right) - g\left(h(\phi X,Z),\phi h_v(X,Z)\right) \\ &= (X^2\ln f)\, g(Z,Z) + (X\ln f)^2 g(Z,Z) - \| h_v(X,Z)\|^2, \end{aligned}$$

where $h_\nu(X,Z)$ denotes the ν-component of $h(X,Z)$. Also, we have

$$\begin{aligned} g(h(\nabla_X \phi X,Z),\phi Z) &= g\left(\bar{\nabla}_Z \nabla_X \phi X,\phi Z\right) \\ &= g\left(\bar{\nabla}_Z \bar{\nabla}_Z \phi X,\phi Z\right) - g\left(\bar{\nabla}_Z h(X,\phi X),\phi Z\right) \\ &= -g(X,X)\, g(Z,Z) + \left((\bar{\nabla}_X X)\ln f\right) g(Z,Z) \\ g\left(h(\phi X,\nabla_X Z),\phi Z\right) &= (X\ln f)\, g\left(h(\phi X,Z),\phi Z\right) = (X\ln f)^2 g(Z,Z) \end{aligned}$$

Substituting the above relation in (2.12) we have

$$\begin{aligned} \bar{R}(X,\phi X,Z,\phi Z) &= 2\|(h_V(X,Y))\|^2 - (X^2\ln f)\, g(Z,Z) \\ &\quad + ((\nabla_X X)\ln f)\, g(Z,Z) - 2g(X,X)\, g(Z,Z) \\ &\quad + ((\phi X)^2 \ln f)\, g(Z,Z) + \left(\left(\nabla_{\phi X}\phi X\right)\ln f\right) g(Z,Z). \end{aligned} \tag{2.13}$$

Now by summing the equation (2.13) and using (2.11) we get

$$\sum_{i=1}^{2\alpha}\sum_{t=1}^{\beta} \| h_v(X,Z)\|^2 = 2\alpha\beta\left(\frac{c+u^2}{2}+1\right) - \beta\Delta(\ln f) \tag{2.14}$$

Next, inequality (2.3) follows from (2.10) and (2.14).

Let h'' be the second fundamental form of M_2 in M. Then, we get

$$g(h''(Z,W),X) = g\left(\nabla_Z W,X\right) = -\left(X\ln f\right) g(Z,W),$$

or equivalently

$$h''(Z,W) = -g(Z,W)\,\nabla(\ln f). \tag{2.15}$$

If the equality sign of (2.3) hold identically then we obtain

$$h(D,D) = 0,\, h(D^{\perp},D^{\perp}) = 0,\, h(D,D^{\perp}) \subset \phi D^{\perp} \tag{2.16}$$

The first condition of (2.16) implies that M_1 is totally geodesic on M. On the other hand, we have

$$g(h(X,\phi Y),\phi Z) = g(\bar{\nabla}_X \phi Y,\phi Z) = g(\bar{\nabla}_X Y,Z) = 0.$$

Thus M_1 is totally geodesic in $\bar{M}(c)$ and hence is a locally conformal almost cosymplectic manifold with constant ϕ-sectional curvature c. The second condition of

(2.16) and (2.15) imply that M_2 is a totally umbilical in $\bar{M}(c)$. Moreover, by (2.16), it follows that M is minimal submanifold of $\bar{M}(c)$.

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