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Continuity on a dense subset of a Baire space

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Abstract: It is proved, in particular, that every real valued quasi-continuous mapping on a Baire space X is continuous on a dense subset of X ; furthermore, we proved that every real valued mapping on a hyperconnected Baire space X is continuous on a dense subset of X if each point of discontinuity is of d_2-type. In fact, in the above results, the range space R of real numbers may be generalised to any secod countable space.

Key words and phrases: Quasi-continuity, Baire space, Hyperconnected space, d_1-point, d_2-point.

1. Introduction:

This note stems from the following theorem of Lin [5]:

Theorem 1.1. (Theorem 1 of [5]). *If $f : X \to Y$ is a mapping from a Baire space X to a second countable space Y, then the mapping f is almost continuous on a dense subset of X.*

Although it is easy to observe that under the hypothesis of the above theorem, a mapping may not be continuous on any dense subset of the domain space; and the required example might have been known, but we are unable to cite a source of print. So, we give the following example.

Example: 1.2. Let N be the set of natural numbers and let τ be the topology consists of all sets O such that $O = \phi$ or $O = N$ or $O = \{1, 2, \ldots, n\}$ for some $n\{>1\}$ in N

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and let $\{R, \mathcal{U}\}$ be the usual topological space of real numbers. Let $f : N \to R$ be a mapping defined by $f(1) = 5$, $f(2) = 7$ and $f(n) = n$ otherwise. It is easy to verify that each point of N is a point of discontinuity of f and so, f is not continuous on any dense subset of N; alghough $\{N, \tau\}$ is a Baire space and $(R, \mathcal{U})$ is a second countable space.

It is then a pertinent and natural question as to under what additional restriction on the mapping or on the space under consideration, the mapping will be continuous on a dense subset of the domain space. Before working on this question, we state some known definitions and results which we need in the sequel.

Definition 1.3. ([4],[6]). A mapping $f : X \to Y$, from a topological space to another, is said to be quasi-continuous at $x \in X$ if for any U, V open such that $x \in U$ and $f(x) \in V$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$; the mapping f is called quasi-continuous on $A \subset X$, if it is quasi-continuous at every point $x \in A$. Every continuous mapping is quasi-continuous, but the converse is not necessarily true.

Definition 1.4. [8]. The boundary of a set A in a topological space X is the set A sans its interior and is denoted by $Bd\,A$. The interior and the closure of a set A in X is denoted by $Int\,A$ and $Cl\,A$ respectively.

Definition 1.5. [1]. A Baire space is a topological space in which the intersection of each countable family of open dense subsets is dense. Every non-empty Baire space is a set of the second category, i.e., it is not the union of a countable family of sets E_n such that $\text{Int}\, ClE_n = \phi$ where ϕ denotes the empty set.

Definition 1.6. [2]. Let $f : X \to Y$ be a mapping from a topological space to another. A point $x \in X$ is called a d_1-point (or a point of d_1-type) of f, if there exists an open neighbourhood N of $f(x)$ such that $x \in B\,d f^{-1}(N)$ and $\text{Int}\, f^{-1}(N) = \phi$; and a point $x \in X$ will be called a d_2-point (or a point of d_2-type) of f, if for any open neighbourhood $N$of $f(x)$, there exists an open sub-neighbourhood O of $f(x)$ [i.e, O is a neighbourhood of $f(x)$ and $O \subset N$] such that $x \in Bdf^{-1}(O)$ and $\text{Int}\, f^{-1}(O) \neq \phi$. If each of these points exists then the set of points of discontinuity is partitioned by the set of d_1-points and the set of d_2-points.

Definition 1.7. [7]. A topological space X is called hyperconnected if every pair of non-empty open sets of X has non-empty intersection, or equivalently, every non-empty open set in X is dense in X.

Now, we have come up with the following result.

Theorem 1.8. If $f : X \to Y$ *is quasi-continuous mapping from a Baire space* X *to a second countable space* Y, *then the mapping* f *is continuous on a dense subset of* X.

We further observe that the assumption 'f *is quasi-continuous*' can be dropped from Theorem 1.8 if the domain space in addition is hyperconnected provided that each point of discontinuity of f is of d_2-type. Precisely, we have.

Theorem 1.9. If $f : X \to Y$ *is mapping from a hyperconnected Baire space* X *to a second countable space* Y *such that the points of discontinuity of* f *(if any) are of* d_2*-type, then* f *is continuous on a dense subset of* X.

The proofs of Theorem 1.8 and Theorem 1.9 are given in the next section. A particular interesting special case of Theorem 1.8 (Theorem 1.9) is obtained by using the usual space R of real numbers in place of the space Y in Theorem 1.8 (Theorem 1.9). Thus

Corollary 1.10. *Every real valued quasi-continuous mapping on a Baire space* X *is continuous on a dense subset of* X.

Corollary 1.11. *Every real valued mapping on hyperconnected Baire space* X *is continuos on a dense subset of* X *if each point of discontinuity of* f *is of* d_2*-type.*

We conclude this section by demonstrating that the concepts of Baire space and hyperconnected space are independent with the help of the followng examples.

Example 1.12. The topological space (N, τ) of Example 1.2 is a Baire space as well as a hyperconnected space.

Example 1.13. Let N be the set of positive integers and τ^* be the cofinite topology. Then (N, τ^*) is a hyperconnected space but not a Baire space.

Example 1.14. The usual space R of real numbers is an obvious example of a Baire space which is not hyperconnected.

2. Proofs of main theorems

Before proving Theorem 1.8, we shall need the following lemma

Lemma 2.1. *A mapping $f : X \to Y$ is quasi-continuous at $x \in X$ if and only if $x \in Cl\ Int\ f^{-1}(V)$ for every open set V containing $f(x)$.*

Proof: First we suppose that f is quasi-continuous at $x \in X$. Let U and V be open such that $x \in U$ and $f(x) \in V$. Then there is a non-empty open set $G \subset U$ such that $f(G) \subset V$. So, $\phi \neq G \subset \text{Int } f^{-1}(V)$. Now, if $x \in G$ then $x \in Cl \text{ Int } f^{-1}(V)$. Again, if $x \notin G$ then $\phi \neq G \subset (U \setminus \{x\}) \cap \text{Int } f^{-1}(V)$ and hence $x \in Cl \text{ Int } f^{-1}(V)$.

Next, let $x \in Cl \text{ Int } f^{-1}(V)$ for every open set V containing $f(x)$. Let U be any open set containing x. Then $U \cap \text{Int } f^{-1}(V) \neq \phi$. We take $G = U \cap \text{Int } f^{-1}(V)$. Hence G is a non-empty open set such that $G \subset U$ and $f(G) \subset f f^{-1}(V) \subset V$.

Proof of Theorem: 1.8. Let $B = \{B_n : n = 1, 2, \ldots\}$ be a countable basis for the open sets in Y. We select $B^* = \{B_{n_1}, B_{n_2}, \ldots\} \subset B$ such that $\text{Int } f^{-1}(B_{n_k}) \neq \phi$, $k = 1, 2, \ldots$. This sub-class B^* is non-empty ; for, if $x \in X$ then by the above lemma $x \in Cl \text{ Int } f^{-1}(B_n)$ for each $B_n \in B$ ontaining $f(x)$ because f is quasi-continuous, and so, $\text{Int } f^{-1}(B_n) \neq \phi$. Now, for each $B_{n_k} \in B^*$, let us set $E_{n_k} = Cl \text{Int } f^{-1}(B_{n_k}) \setminus \text{Int } f^{-1}(B_{n_k})$. Then $\text{Int } Cl\ E_{n_k} = \text{Int } E_{n_k} = \text{Int } Cl \text{ Int } f^{-1}(B_{n_k}) \setminus Cl \text{Int} f^{-1}(B_{n_k}) = \phi$ for such k, and thus the set $E = \cup_{k=1}^{\infty} E_{n_k}$ is a set of the first category. But if f is not continuous at x, then there exists a $B_{n_k} \in B^*$ such that $f(x) \in B_{n_k}$ and $x \notin \text{Int } f^{-1}(B_{n_k})$. But since f is quasi-continuous at $x, x \in Cl \text{ Int } f^{-1}(B_{n_k})$ by the lemma stated earlier. Hence $x \in E_{n_k}$ for some k and so, $x \in E$. Therefore, f is continuous on $X \setminus E$, which as a complement of a first category subset of a Baire space, is dense in X.

Proof of Theorem 1.9. Let $B = \{B_n : n = 1, 2, \ldots\}$ be a countable basis for the open sets in Y. We select $B^* = \{B_{n_1}, B_{n_2}, \ldots\} \subset B$ such that $\text{Int } f^{-1}(B_{n_k}) \neq \phi$, $k = 1, 2, \ldots$ This sub-class B^* is non-empty, because if x is a point of discontinuity then x is a d_2-point and hence there exists a $B_n \in B$ such that $x \in Bd\, f^{-1}(B_n)$ and $\text{Int } f^{-1}(B_n) \neq \phi$. Now, for each $B_{n_k} \in B^*$, we set $E_{n_k} = f^{-1}(B_{n_k}) \setminus \text{Int } f^{-1}(B_{n_k})$. Since X is hyperconnected, for each $B_{n_k} \in B^*$, $Cl \text{ Int } f^{-1}(B_{n_k}) = X$. So, for each k, $\text{Int } Cl\ E_{n_k} \subset \text{Int } Cl\, f^{-1}(B_{n_k}) \setminus Cl \text{ Int } f^{-1}(B_{n_k}) = \phi$, i.e., E_{n_k} is nowhere dense for each k ; and thus the set $F = \cup_{k=1}^{\infty} E_{n_k}$ is a set of the first category. But if f is not continuous at x, there exists a $B_{n_k} \in B^*$ such that $f(x) \in B_{n_k}$ and that $x \notin \text{Int}\, f^{-1}(B_{n_k})$, consequently, $x \in E_{n_k}$ because $x \in f^{-1}(B_{n_k})$. Hence $x \in E$, and so, f is continuous on the dense subset $X \setminus E$ of X.

Remark 2.2. The condition '*f is quasi-continuous on X*' in Theorem 1.8 is sufficient but not a necessary one as shown in the following example.

Example 2.3. Consider the topological spaces (N, τ) and $(R, \mathcal{U})$ of Example 1.2. Let $f : N \to R$ be a mapping defined by $f(2) = 1$ and $f(n) = n$ otherwise. Clearly, f is not quasi-continuous at $n(\neq 1, 2)$; but f is continuous on the dense subset $\{1,2\}$ of N.

Remark 2.4. Since each point of discontinuity of f, viz., $n(\neq 1,2)$ of Example 2.3 is a d_1-point, it is clear that the hypothesis '*points of discontinuity of f (if any) are of d_2-type*' of Theorem 1.9 is only a sufficient condition but not a necessary one.

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