The Nepali Math. Sc. Report
Vol. 29, No. 1 & 2, 2009

DCP Property of a Certain Combinations of de la Vallée Poussin Kernels

CHINTA MANI POKHAREL
Nepal Engineering College
G.P.O. Box 10210 Kathmandu, Nepal
email: chintam@nec.edu.np

Abstract: We shall established the DCP Property of certain Combinations of de la Vallée Poussin Kernels for some particular cases.

1. INTRODUCTION

Let $\mathcal{A}$ denote the set of analytic functions in D, $f * g$ the Hadamard product or convolution between two members of $\mathcal{A}$. A domain $\Omega \subseteq C$ is said to be *convex in the direction* $e^{i\phi}$, $\phi \in \Re$, if and only if for every $a \in C$ the set.

$$\Omega \cap \{a + te^{i\phi} : t \in \Re\}$$

is either connected or empty. Accordingly we define the class $\mathcal{K}(\phi) \subset \mathcal{A}$, $\phi \in \Re$, of the functions *convex in the direction* $e^{i\phi}$ as

$$\mathcal{K}(\phi) := \{f \in \mathcal{A} : f \text{ univalent and } f(D) \text{ convex in the direction } e^{i\phi}\}.$$

Finally, a function $g \in \mathcal{A}$ is called *Direction-Convexity-Preserving* ($g \in$ DCP) if and only if

$$g * f \in \mathcal{K}(f) \text{ for all } f \in \mathcal{K}(\phi) \text{ and all } \phi \in \Re.$$

Functions in DCP have many other intriguing convolution-type properties, for instance the preservation of convex harmonic functions in D, and of Jordan curves in the plane with convex interior domain; we refer to [7], [8] for more details. There one also finds a complete description of the members of DCP, namely

$$g \in DCP \Longleftrightarrow g(z) + itzg'(z) \in \not\prec (\frac{\pi}{2}) \text{ for all } t \in \Re.$$

Further it is known, that DCP functions are convex univalent.

The following criterion for membership in DCP is a slight variant of [7, Theorem 4].

Lemma 1 Let g be analytic in $\bar{D}$, convex univalent and let $u(t) := \text{Re}g(e^{it}), t \in \Re$. Then

$g \in DCP$ if and only if

$$\sigma_u := (u''(t))^2 - u'(t)\, u'''(t) \geq 0,\ t \in \Re$$

The classical definition of the de la Vallée Poussin Kernel of order $n \in N$ is

$$w_n(t) := \frac{2^n (n!)^2}{(2n)!}(1 + \cos(t))^n \tag{1}$$

$$= \frac{1}{\binom{2n}{n}} \sum_{k=-n}^{n} \binom{2n}{n+k} e^{ikt}.$$

But here we are interested in the analytic version of the de la Vallée Poussin Kernel

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k,\ z \in C. \tag{2}$$

Note that

$$2\text{Re}\, V_n(e^{it}) = w_n(t) - 1,\ n \in N. \tag{3}$$

2. MAIN RESULTS

In this section, we again come back to the analytic version of the classical de la Vallee Poussin Kernels. Let us recall that the function

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k,\ z \in D,$$

is the de la Vallee Poussin kernel of order n.

In [9], St. Ruscheweyh and J. K. Wirths proved that for $0 < x < \infty$ and for $n \in N$, the function

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k} x^k \binom{2k}{k} V_k(z), z \in D, \tag{4}$$

is convex. When they proved this, the class DCP was not even defined. Later in 1989 [7], St. Ruscheweyh and L. Salinas introduced the class DCP, which is a sublcass of the class of convex functions. Now one can ask a natural question whether the functions $f_n(z)$ belong to the class DCP instead of just to the class of convex functions. We shall prove that in general the function $f_n(z)$ does not belong to the class DCP. Already for the special case $x = 1$ in (4), we get the following result.

Theorem 1 For $n \in N$, let

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} V_k(z), \ z \in D$$

Then $f_n \in$ DCP for $n \leq 6$ and $\notin$ DCP for $n = 7$.

3. PROOF OF THE MAIN RESULT

Proof: Just like in the previous sections, put

$$w_k(t) := \mathrm{Re}\, V_k(e^{it}) = -\frac{1}{2} + \frac{2^{k-1}(k!)^2}{(2k)!}(1+\cos(t))^k$$

and let

$$\begin{aligned} u_n(t) := \mathrm{Re}\, f_n(e^{it}) &= \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} w_k(t) \\ &= \sum_{k=1}^{n} \binom{n}{k} \binom{2k}{k} \left(-\frac{1}{2} + \frac{2^{k-1}(k!)^2}{(2k)!}(1+\cos(t))^k\right) \\ &= \sum_{k=1}^{n} \left(-\frac{n!\,(2k)!}{2(n-k)!\,(k!)^3} + \frac{2^{k-1}\, n!}{k!\,(n-k)!}(1+\cos(t))^k\right) \end{aligned}$$

Then, from lemma 1, $f_n \in$ DCP if and only if

$$v_n(t) := u''_n(t)\, u''_n(t) - u'''_n(t)\, u'_n(t) \geq 0 \text{ for } 0 \leq t \leq 2n.$$

After simplification (using Mathematica 3.0), we get:

$$
\begin{aligned}
v_1(t) &= 1,\\
v_2(t) &= 52 + 54\cos(t) - 6\cos(3t)\\
v_3(t) &= 9\,(3 + 2\cos(t))^2\,(15 + 15\cos(t) - 2\cos(2t) - 3\cos(3t)),\\
v_4(t) &= 8\,(3 + 2\cos(t))^4\,(34 + 33\cos(t) - 8\cos(2t) - 9\cos(3t)),\\
v_5(t) &= 25\,(3 + 2\cos(t))^6\,(19 + 18\cos(t) - 6\cos(2t) - 6\cos(3t)),\\
v_6(t) &= 18\,(3 + 2\cos(t))^8\,(42 + 39\cos(t) - 16\cos(2t) - 15\cos(3t)),\\
v_7(t) &= 49\,(3 + 2\cos(t))^{10}\,(23 + 21\cos(t) - 10\cos(2t) - 9\cos(3t)).
\end{aligned}
$$

We shall show one by one that

$$u_n(t) \geq 0,\ 0 \leq t \leq 2\pi, \tag{5}$$

for $n \leq 6$, while $v_7(t)$ does not satisfy this condition.

The case $n = 1$ is obvious. For the case $n = 2$,

$$v_2(t) = 52 + 54\cos(t) - 6\cos(3t) = 52 + 72x - 24x^3 =: p_2(x)$$

where $x = \cos(t)$. Therefore $v_2(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if the polynomial $p_2(x) \geq 0$ on $-1 \leq x \leq 1$. Now for $-1 \leq x \leq 1$,

$$
\begin{aligned}
p_2(x) &= 52 + 24x(3 - x^2)\\
&\geq 52 + 48x\\
&\geq 4.
\end{aligned}
$$

Therefore (5) holds for the case $n = 2$.

Now consider the case $n = 3$. After a simple calculation, we can write

$$v_3(t) = 9(3 + 2\cos(t))^2\, p_3(x), \tag{6}$$

where

$$p_3(x) = 17 + 24x - 4x^2 - 12x^3 \text{ and } x = \cos(t).$$

From (6), we see that $v_3(t) \geq 0$ for $0 \leq t \leq 2\pi$ if and only if $p_3(x) \geq 0$ for $-1 \leq x \leq 1$. Now

$$p_3(-1) = 1,\ p_3(1) = 25,$$

while

$$p_3(x_1) \approx 0.871904,\quad p_3(x_2) \approx 27.7289$$

at the critical points

$$x_1 = \frac{1}{9}(-1 - \sqrt{55}) \approx -0.935133,\ x_2 = \frac{1}{9}(-1 + \sqrt{55}) \approx 0.712911,$$

both of which lie inside the interval $[-1, 1]$. This shows that

$$p_3(x) \geq p_3(x_1) > 0 \text{ for } -1 \leq x \leq 1,$$

and hence $v_3(t) \geq 0$ for $0 \leq t \leq 2\pi$.

Consider the case n = 4. As in the case of $v_3(t)$, we can write

$$v_4(t) = 8\,(3 + 2\cos(t))^4\, p_4(x), \tag{7}$$

where

$$p_4(x) = 42 + 60x - 16x^2 - 36x^3 \text{ and } x = \cos(t)).$$

It is clear from (7) that $v_4(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_4(x) \geq 0$ on $-1 \leq x \leq 1$. If we study the behaviour of the polynomial $p_4(x)$ on $[-1, 1]$, we see that

$$p_4(-1) = 2,\ p_4(1) = 50,$$

and for $x \in (-1, 1)$, $p_4(x)$ has critical points at $x_1 = \frac{1}{27}(-4 - \sqrt{421}) \approx -0.908085 > -1$ and $x_2 = \frac{1}{27}(-4 + \sqrt{421}) \approx 0.611788 < 1$, and $p_4(x)$ takes positive values at both of these points. In fact, $p_4(x_1) \approx 1.27865$ and $p_4(x_2) \approx 64.4753$. From this we conclude that $p_4(x) \geq 0$ on $-1 \leq x \leq 1$, and hence $v_4(t) \geq 0$ on $0 \leq t \leq 2\pi$.

For the case n = 5, we can write

$$v_5(t) = 25\,(3 + 2\cos(t))^6\, p_5(x), \tag{8}$$

where

$$p_5(x) = 25 + 36x - 12x^2 - 24x^3 \text{ and } x = \cos(t).$$

We see here also that $v_5(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_5(x) \geq 0$ on $-1 \leq x \leq 1$. Now

$$p_5(-1) = 1,\quad p_5(1) = 25,$$

and for $x \in (-1, 1)$, $p_5(x)$ has critical points: one at $x_1 = \frac{1}{6}(-4 - \sqrt{19}) \approx -0.89315 > -1$ and the other at $x_2 = \frac{1}{6}(-1 + \sqrt{19}) \approx 0.559816 < 1$. $p_5(x)$ takes positive values at both of these points; in fact, $p_5(x_1) \approx 0.373538$ and $p_5(x_2) \approx 37.182$. From this we conclude that $p_5(x) \geq 0$ on $-1 \leq x \leq 1$, and hence $v_5(t) \geq 0$ on $0 \leq t \leq 2\pi$.

Case n = 6. As in the previous case, let us write

$$v_6(t) = 18\,(3 + 2\cos(t))^8\, p_6(x), \tag{9}$$

where

$$p_6(x) = 58 + 84x - 32x^2 - 60x^3 \text{ and } x = \cos(t).$$

If we study the behaviour of $p_6(x)$ on $[-1, 1]$, we see that $p_6(-1) = 2$, $p_6(1) = 50$. And for $x \in (-1, 1)$, $p_6(x)$ has critical points at $x_1 = \frac{1}{45}(-8 - \sqrt{1009}) \approx -0.883661$ and $x_2 = \frac{1}{45}(-8 + \sqrt{1009}) \approx 0.528106$, and $p_6(x)$ takes positive values at both of these points. In fact, $p_6(x_1) \approx 0.18582$ and $p_6(x_2) \approx 84.599$. We thus see that $p_6(x) \geq 0$ on $-1 \leq x \leq 1$, and hence, from (9), we conclude that $v_6(t) \geq 0$ on $0 \leq t \leq 2\pi$. In this way we have shown that the functions $v_n(t) \geq 0$ on $0 \leq t \leq 2\pi$ for $n = 1, ..., 6$. and hence the functions

$$f_n(z) = \sum_{k=1}^{n} \binom{n}{k}\binom{2k}{k} V_k(z)$$

are in the class DCP for $n \in N$, $n \leq 6$.

Moving on to the case $n = 7$, we now show that the condition $v_7(t) \geq 0$ for $0 \leq t \leq 2\pi$ does not hold. After a simple calculation, we can write

$$v_7(t) = 49\,(3 + 2\cos(t))^{10}\, p_7(x) \tag{10}$$

where

$$p_7(x) = 33 + 48x - 20x^2 - 36x^3 \text{ and } x = \cos(t).$$

Here also, we see that $v_7(t) \geq 0$ on $0 \leq t \leq 2\pi$ if and only if $p_7(x) \geq 0$ on $-1 \leq x \leq 1$. But for this case, we have

$$p_7(-1) = 1, \quad p_7(1) = 25$$

and

$$p_7(x_1) \approx -0.19564, \; p_7(x_2) \approx 47.5034$$

at the critical points $x_1 = \frac{1}{27}(-5 - \sqrt{349}) \approx -0.877094 > -1$ and $x_2 = \frac{1}{27}(-5 + \sqrt{349}) \approx 0.506724$, both of which lie inside the closed interval $[-1, 1]$. This shows that $p_7(x)$ takes also negative values in $-1 \leq x \leq 1$ and consequently $v_7(t)$ takes also negative values in $0 \leq t \leq 2\pi$. Hence f_n can not belong to the class DCP for $n = 7$. This completes the proof of this case and of the theorem as well.

REFERENCES

1. S.D. Bernardi, (1961). *Convex, starlike, and level curves,* Duke Math. J. 28, 57-72.
2. J.E. Brown, (1987). *Level sets for functions convex inone directions,* Proc. Amer. Math. Soc. 100, 442–446.
3. W. Hengratner and G. Schober, (1973). *A remarks on level curves for domains convex in one directions,* Applicable Anal. 3, 101 – 106.

4. W. Hengratner and G. Schober, (1970). *On schlicht mappings to domains convex in one directions,* Comment. Math. Helv. 45, 303–314.

5. A.W. Goodman and E.B. Saff, (1979). *On univalent functions convex in one direction,* Proc. Amer. Math. Soc. 73, 183–187.

6. G. Polya and I. J. Schoenberg, (1958). *Remarks on the de la Vallee Poussian mean and convex conformal maps of the circle,* Pacific J. Math. 8, 295–333.

7. St. Ruscheweyh and L. Salinas, (1989). *On the preservation of direction convexity and the Goodman – Saff conjecture,* Ann. Acad. Sci. Fenn. 14, 63 – 73.

8. St. Ruscheweyh and L. Salinas, (1992). *On the preservation of periodic Monotonicity,* Constr. Approx. 8, 128 – 140.

9. St. Ruscheweyh and K. J. Wirths, (1976). *Riemann mapping theorem for n-analytic functions,* Math. Z. 149, 287–297.