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Derivation of a class of bilateral generating functions from a set of orthogonal polynomials

N. K. RANA

Abstract: A new class of bilateral generating functions are obtained from a set of orthogonal polynomials $\{F^m_{mn+v}(x\,;\lambda v)\}$. Known bilateral generating functions are obtained as a particular case.

1. Introduction

In 1983, Datta and Manocha [2] introduced a new class of orthogonal Polynomials $\{F^m_{mn+v}(x\,;\lambda\, v)\}$ with the help of a relation

$$(1.1)\qquad F^m_{mn+v}(x\,;\lambda v) = x^v\,{}_1F_1\frac{(-n;2v+\lambda+m-1\,;\,x^m)}{m}$$

The polynomials $\{F^m_{mn+v}(x\,;\lambda v)\}$ satisfies the following differential equation

$$(1.2)\qquad x^2y'' - (mx^{m+1}-\lambda x)y' + \{m(mn+v)x^m - v(v+\lambda-1)\}y = 0$$

where λ, m are fixed parameters, n is a variable parameter and v a non-negative integer $< m$. Aim of the present paper is to obtain a new class of bilateral generating functions for $\{F^m_{mn+v}(x\,;\lambda\, v)\}$ which can be written in the form of the following theorem:

Theorem: *If there exists a unilateral generating function of the form*

$$(1.3)\qquad G(x,t) = \sum_{n=0}^{\infty} a_n F^m_{mn+v}(x\,;\lambda v)\, t^n$$

then the following class of bilateral generating function will hold:

$$(1.4)\qquad (1-mty)^{-(m+v+\lambda-1)/m}\exp(-mx^m ty/(1-mty))\,.\,f\,[x/(1-mty)^{1/m}, ty/(1-mty)]$$

$$= \sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n m^{p-n}/(p-n)!(n+(2v+\lambda-1+m)/m)_{p-n}\, F^m_{mp+v}(x;\lambda\, v)(ty)^p$$

The importance of the theorem lies on the fact that all particular bilateral. Generating functions can be easily deduced by attributing suitable values to a_n and then making use of known linear generating functions involving the class of orthogonal polynomials.

2. Derivation of the generating functions

For $\{F^{m}_{mn+v}(x;\lambda v)\}$ we have the following partial differential operator [6]

(2.1) $$B = e^{y}\{x\partial/\partial x + m\partial/\partial y + (m+v+\lambda-1-mx^{m})\}$$

such that

(2.2) $$B[F^{m}_{mn+v}(x;\lambda v)e^{ny}] = \{m(n+1)+2v+\lambda-1\}F^{m}_{m(n+1)+v}(x;\lambda v)e^{(n+1)y}$$

The extended form of the transformation group generated by B is

(2.3) $$[T(\exp(bB)f](x,t) = (1-mtb)^{-(m+v+\lambda-1)/m}\exp(-mx^{m}tb/(1-mtb)).\ f[x/(1-mtb)^{1/m}, t/(1-mtb)]$$

Let

(2.4) $G(x,y) = \sum_{n=0}^{\infty} a_n F^{m}_{mn+v}(x;\lambda v)y^{n}$, where a_n is arbitrary, selected in such a way that the left hand side of (2.4) gives known generating functions.

Replacing y by ty in (2.4)

(2.5) $$G(x,ty) = \sum_{n=0}^{\infty} a_n F^{m}_{mn+v}(x;\lambda v)t^{n}y^{n}$$

Operating exp (bB) on both sides of (2.5), left hand side becomes

$$(1-mtby)^{-(m+v+\lambda-1)/m}\exp(-mx^{m}tby/(1-mtby)).f[x/(1-mtby)^{1/m}, ty/(1-mtby)]$$

On the other hand, right hand side reduces to

$$\sum_{p=0}^{\infty}\sum_{n=0}^{p} a_n b^{p-n}m^{p-n}/(p-n)!\{(n+(2v+\lambda-1+m)/m)_{p-n} F^{m}_{mn+v}(x;\lambda v)(ty)^{p}$$

Equating and putting $b = 1$, we obtain (1.4).

3. Applications

Assuming $a_n = (-p)_n(-n)_n/p!$ and using $f(x,y) = y^{n}F^{m}_{mn+v}(x;\lambda v)$

We get

(3.1) $$(1-mty)^{-(m+v+\lambda-1)/m}\exp(-mx^{m}ty/(1-mty)).x^{v}/(1-mty)^{n+v/m}.$$
$$ {}_1F_1(-n;(2v+\lambda-1+m)/m;x^{m}/(1-mty)) =$$
$$\sum_{p=0}^{\infty}\frac{|(2v+\lambda-1+m/m)|_p}{p!}\,{}_2F_1(-p,-n;(2v+\lambda-1+m)/m;1/mt)F^{m}_{mp+v}(x;\lambda v)(mt)^{p}(y)^{p-n}$$

Changing my by y and putting $t = 1$, we obtain the bilateral generating relation

(3.2) $$(1-y)^{-(n+m+2v+\lambda-1)/m}\exp(-x^{m}y/(1-y)).x^{v} \times {}_1F_1(-n;(2v+\lambda-1+m)/m;x^{m}/(1-y))$$
$$= \sum_{p=0}^{\infty}\frac{((2v+\lambda-1+m)/m)_p}{p!}\,{}_2F_1(-p,-n;(2v+\lambda-1+m)/m;1/mt)\,F^{m}_{mp+v}(x;\lambda v)$$

This was derived by Bhattacharya and Rath [1]

Putting m = 1, v = 0 and λ = 1 + α and nothing that $F_n'(x;1+\alpha,0) = n!(1+\alpha)_n L_n^{(\alpha)}(x)$ the bilateral generating relation (3.2) reduces to

$$(1-y)^{-n-\alpha-1}\exp(-xy/(1-y))\,{}_1F_1(-n;1+\alpha;x/(1-y)) =$$
$$\sum_{p=0}^{\infty} {}_2F_1(-p,-n;1+\alpha;1)\,L_p^{\alpha}(x)y^{p-n}$$

which can be compared with a result of Mcbride (p.37)

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N. K. RANA

Kalora High School (H.S)
P.O. Kalora, Dist. Midnapur
Pin.– 721146.