

## ERMAKOV EQUATION AND CAMASSA-HOLM WAVES

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**Abstract:** Since the works of [1] and [2], it is known that the solution of the Ermakov equation is an important ingredient in the spectral problem of the Camassa-Holm equation. Here, we review this interesting issue and consider in addition more features of the Ermakov equation which have an impact on the behavior of the shallow water waves as described by the Camassa-Holm equation.

**Key Words:** Ermakov equation, Camassa-Holm equation, nonlinear waves

**AMS (MOS) Subject Classification.** 34A05, 34A34, 35Q35.

### 1. INTRODUCTION

The Camassa-Holm equation ([3])

$$(1.1) \quad u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

can be factored ([4]) as

$$(1.2) \quad \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + 2u_x \right) q(x, t) = 0, \quad q(x, t) = (u - u_{xx} + k),$$

where  $q(x, t)$  is known as momentum. There are two conservation laws

$$(1.3) \quad \frac{\partial}{\partial t} q(x, t) + \frac{\partial}{\partial x} \left[ 2ku + \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}(u_x)^2 \right] = 0,$$

and

$$(1.4) \quad (\sqrt{q})_t + (u\sqrt{q})_x = 0$$

respectively. Using the momentum, Eq. (1.1) is equivalent to

$$(1.5) \quad q_t + uq_x + 2u_x q = 0.$$

The Lax pairs for CH are (Constantin 2001)

$$(1.6) \quad \begin{cases} \psi_{xx} = \left( \frac{1}{4} + \lambda q \right) \psi \\ \psi_t = \left( \frac{1}{2\lambda} - u \right) \psi_x + \frac{1}{2} u_x \psi \end{cases}$$

and the compatibility condition  $\psi_{xxt} = \psi_{txx}$  leads back to Eq. (1.1).

## 2. LIOUVILLE TRANSFORMATION ON CH

The common procedure is to use  $q = \left(\frac{dy}{dx}\right)^2$  followed by Liouville's transformation  $\phi = q^{\frac{1}{4}}\psi$  in the first equation of the Lax pair system (1.6) to obtain

$$(2.1) \quad \phi_{yy} - \left[ \frac{4q(1 + q_{yy}) - 3(q_y)^2}{16q^2} \right] \phi = \lambda\phi.$$

In addition, since  $q(x, t) = u - u_{xx} + k$  with  $\frac{dy}{dx} = \sqrt{q}$ , the momentum equation becomes

$$(2.2) \quad qu_{yy} + \frac{1}{2}q_y u_y - u = k - q$$

Therefore, if we know  $q$  then one can try to solve CH by finding  $u$  in Eq. (2.2). As a scattering Schrödinger problem, we can rewrite Eq. (2.1) as

$$(2.3) \quad \phi_{yy} - \left[ Q + \frac{1}{4k} \right] \phi = \lambda\phi$$

with potential  $Q$  defined in terms of the momentum by

$$(2.4) \quad q_{yy} - \frac{3}{4q}(q_y)^2 - 4 \left[ Q + \frac{1}{4k} \right] q + 1 = 0$$

Let  $q = E^4$ , then we obtain the Ermakov equation

$$(2.5) \quad E_{yy} - \left[ Q + \frac{1}{4k} \right] E + \frac{1}{4}E^{-3} = 0$$

The solution of Eq. (2.5) is given by Pinney (1950)

$$(2.6) \quad E = \sqrt{F_1^2 - \frac{1}{4} \left( \frac{F_2}{W} \right)^2}$$

where  $F_1$  and  $F_2$  are the two independent solutions of the linear ODE

$$(2.7) \quad F_{yy} - \left[ Q + \frac{1}{4k} \right] F = 0$$

The Ermakov-Lewis invariant is

$$(2.8) \quad \mathcal{I} = \frac{1}{2} \left[ (EF_y - FE_y)^2 - \frac{1}{4} \left( \frac{E}{F} \right)^{-2} \right].$$

By using  $E$  from Eq. (2.5) and  $F$  as a superposition of homogenous solutions  $F = aF_1 + bF_2$ , the invariant is

$$(2.9) \quad \mathcal{I} = \frac{1}{2} \left( -\frac{a^2}{4} + b^2 W^2 \right) = \text{const.}$$

It is easy to show that Eq. (2.7) and (2.5) are related to the linear third order ODE

$$(2.10) \quad \phi_{yyy} - 4 \left[ Q + \frac{1}{4k} \right] \phi_y - 2Q_y \phi = 0,$$

which is of maximal symmetry algebra  $sp(5)$  ([5]) and has  $\phi$  itself as an integrating factor; thus Eq. (2.10) becomes

$$(2.11) \quad \phi\phi_{yy} - \frac{1}{2}(\phi_y)^2 - 2 \left[ Q + \frac{1}{4k} \right] \phi^2 + \frac{1}{2} = 0.$$

## 3. SOLUTIONS

Let us use the solitonic potential

$$(3.1) \quad Q = -2K^2 \operatorname{sech}^2(\theta), \quad \theta = Ky + \Omega t - \alpha$$

where  $K = \frac{\mu}{\sqrt{k}}$ ,  $\Omega = \frac{\mu}{2\lambda}$ ,  $\mu = \frac{1}{2}\sqrt{1+4\lambda k}$ , and  $\alpha$ , an arbitrary phase. Thus, Eq. (2.10) becomes

$$(3.2) \quad \phi_{\theta\theta\theta} + 4 \left[ 2 \operatorname{sech}^2\theta - \frac{1}{1 - \frac{2k}{c}} \right] \phi_{\theta} - 8 \operatorname{sech}^2\theta \tanh\theta \phi = 0$$

and has solution

$$(3.3) \quad \phi(\theta) = \frac{c}{2\sqrt{k}} \left( \operatorname{sech}^2\theta + \frac{2k}{c} \tanh^2\theta \right),$$

where  $c = -\frac{1}{2\lambda}$ . Since  $q = \phi^2$ , we get

$$(3.4) \quad q(\theta) = \frac{c^2}{4k} \left( \operatorname{sech}^2\theta + \frac{2k}{c} \tanh^2\theta \right)^2$$

In the  $\theta$  variable Eq. (2.2) becomes

$$(3.5) \quad \frac{1}{4k} \left( 1 - \frac{2k}{c} \right) \left( qu_{\theta\theta} + \frac{1}{2}q_{\theta}u_{\theta} \right) - u = k - q$$

and after we substitute  $q, q_{\theta}$  in Eq. (3.5) it becomes

$$(3.6) \quad [c - k + k \cosh^2(2\theta)]^2 u_{\theta\theta} - 2(c - 2k) [c - k + k \cosh^2(2\theta)] \tanh\theta u_{\theta} - 16ck^2u = f(\theta)$$

with nonhomogenous function given by  $f(\theta) = -(c - 2k) \operatorname{sech}^4\theta [4ck(c + 2k \cosh(2\theta))]$ . For the particular case of  $k \rightarrow 0$  Eq. (3.6) simplifies to

$$(3.7) \quad u_{\theta\theta} - 2 \tanh\theta u_{\theta} = 0$$

with solution

$$(3.8) \quad u(\theta) = \mathcal{C}_1 + \mathcal{C}_2 \left( \frac{\theta}{2} + \frac{1}{4} \sinh 2\theta \right).$$

Solving Eq. (3.6) yields to general solution  $u(\theta) = u_p(\theta) + \mathcal{C}_1 u_1 + i\mathcal{C}_2 u_2$ . Denoting  $k_c = \frac{2k}{c}$  the particular solution is

$$(3.9) \quad u_p(\theta) = \frac{c(1 - k_c)}{1 - k_c + k_c \cosh^2\theta} = \frac{c(1 - k_c)}{1 + k_c \sinh^2\theta}$$

while the general solutions are

$$(3.10) \quad \begin{cases} u_1(\theta) = \cosh \left\{ \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh} [\sqrt{1-k_c} \tanh \theta] \right\}, & 0 < k_c < 1 \\ u_2(\theta) = \sinh \left\{ \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh} [\sqrt{1-k_c} \tanh \theta] \right\}, & 0 < k_c < 1 \end{cases}$$

and

$$(3.11) \quad \begin{cases} u_1(\theta) = \cos \left\{ \frac{2}{\sqrt{k_c-1}}\theta + 2\operatorname{arctan} [\sqrt{k_c-1} \tanh \theta] \right\}, & k_c > 1 \\ u_2(\theta) = -i \sin \left\{ \frac{2}{\sqrt{k_c-1}}\theta + 2\operatorname{arctan} [\sqrt{k_c-1} \tanh \theta] \right\}, & k_c > 1. \end{cases}$$

See Fig. 1 (top) for the particular solution  $u_p$  if  $k_c < 1$  which shows that the soliton profiles are tending to a compacton when  $k_c$  is small, and on (bottom) when  $k_c > 1$  which shows that soliton profiles are more and more peakon-like. In Figs. 2 and 3 we show the

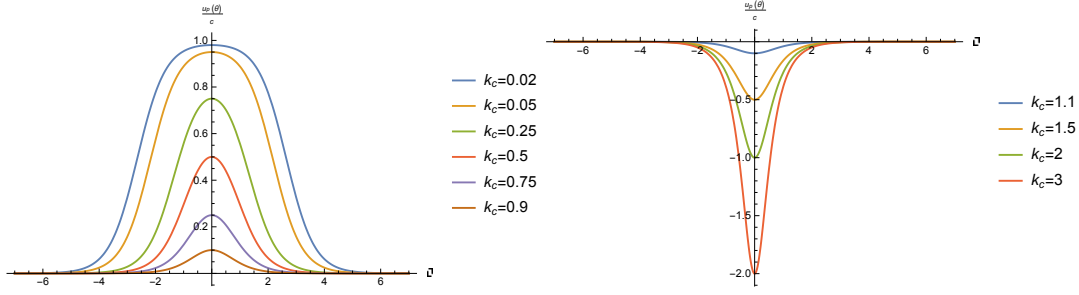


FIGURE 1. The particular solution  $\frac{u_p}{c}$  for  $k_c = \frac{2k}{c} < 1$  (top) and  $k_c = \frac{2k}{c} > 1$  (bottom).

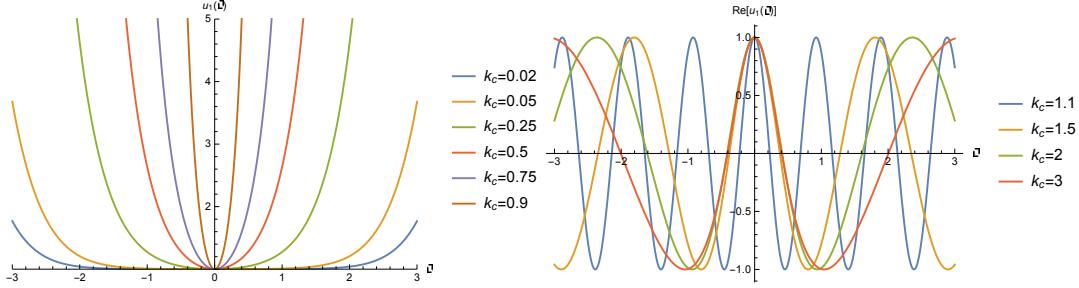


FIGURE 2. The particular solution  $u_1$  for  $k_c = \frac{2k}{c} < 1$  (top) and  $k_c = \frac{2k}{c} > 1$  (bottom).

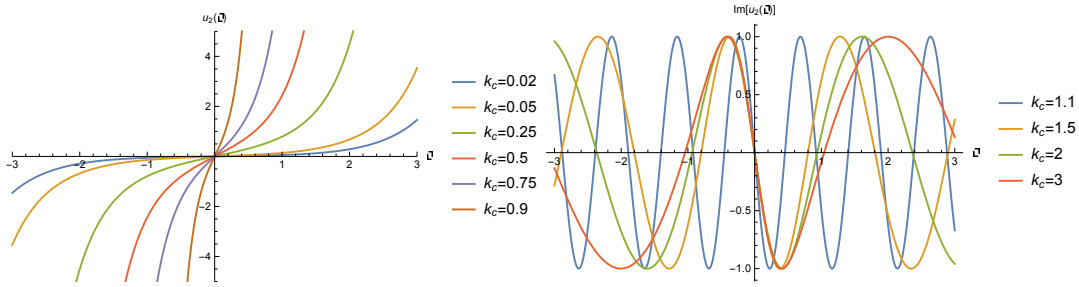


FIGURE 3. Same as in the previous figure for particular solution  $u_2$ .

particular solution  $u_1$  and  $u_2$  for both cases.

To find the solution in terms of  $x$  and  $t$ , we need to use the relation between  $\theta$  and  $x$ . Since  $\sqrt{q(\theta)} = \frac{dy}{dx}$ , we have

$$(3.12) \quad \int \frac{d\theta}{\operatorname{sech}^2\theta + k_c \tanh^2\theta} = \frac{\sqrt{1-k_c}}{2k_c} (x - ct - x_0).$$

To find this integral we notice that if we differentiate the argument of the homogenous solutions  $u_1$  and  $u_2$ , we have

$$(3.13) \quad \frac{d}{d\theta} \left\{ \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh} \left[ \sqrt{1-k_c} \tanh \theta \right] \right\} = \frac{2k_c}{\sqrt{1-k_c}} \frac{1}{\operatorname{sech}^2\theta + k_c \tanh^2\theta}$$

Using Eqs. (3.12) and (3.13) we conclude that

$$(3.14) \quad x - ct - x_0 = \begin{cases} \frac{2}{\sqrt{1-k_c}}\theta - 2\operatorname{arctanh} \left[ \sqrt{1-k_c} \tanh \theta \right], & 0 < k_c < 1 \\ \frac{2}{\sqrt{k_c-1}}\theta + 2\operatorname{arctan} \left[ \sqrt{k_c-1} \tanh \theta \right], & k_c > 1 \end{cases}$$

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