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Fixed Point Theorems in Dislocated Quasi D-Metric Spaces

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Abstract: In this paper we have proved fixed point theorem for continuous contraction mapping in dislocated quasi D-metric space. Also we obtain a common fixed point theorem for pairs of mappings and four mappings in dislocated D-metric space.

Keywords: dislocated quasi D-metric, fixed point.

1 Introduction:

B.E. Rhoades [1] established various definitions of contractive mappings. In 1992 Banach proved fixed point theorem for contraction mapping in complete metric space. In 1992 , Dhage [2] introduced a generalized metric space or D-metric space, and proved the existence of unique fixed point of a self map satisfying a contractive condition. Rhoades [3] generalized Dhage's contractive condition and obtained some fixed point theorems.

In 2005 F.M. Zeyada, G.H. Hassan, M.A. Ahmed [6] established various definitions of dislocated quasi metric space. A Isufati [11] proved some fixed point theorems for a single and a pairs of mappings in dislocated metric space.

In this paper we proved fixed point theorem for continuous contraction mapping in dislocated quasi D-metric space and also proved common fixed point theorems for pairs of mappings and four mappings in dislocated quasi D-metrics space.

2 Preliminaries:

Definition 2.1 Let X be a non-empty set and let $D: X\times X\times X \to [0,\infty)$ be a function satisfying following conditions

(i) $D(x,y,z) = 0 \Leftrightarrow x = y = z$.

(ii) $D(x,y,z) \le D(x,y,a) + D(x,a,z) + D(a,y,z)\ \forall\, x,y,z,a \in X$.

Then D is called a dislocated quasi D-metric on X. If D satisfies $D(x,y,z) = D(y,z,x) = D(z,x,y)$ then it is called dislocated D-metric.

Definition 2.2 A sequence $\{x_n\}$ in dislocated quasi D-metric space (X.D) is called Cauchy sequences if for given $\varepsilon > 0$, $\exists n_0 \in N$ such that $\forall$ $m,n \ge n_0$ $\Rightarrow D(x_m,x_n,x) < \varepsilon$ (or) $D(x_n,x_m,x) < \varepsilon$ i.e $\min\{D(x_m,x_n,x), D(x_n,x_m,x)\} < \varepsilon$.

Definition 2.3 A sequence $\{x_n\}$ in dislocated quasi D-converges to x

$$\lim_{m,n\to\infty} D(x_m,x_n,x) = 0 = \lim_{m,n\to\infty} D(x_n,x_m,x) = 0$$

In this case x is called a dislocated quasi-limit of $\{x_n\}$ and we write $x_n \to x, x_m \to x$.

Lemma 2.4 Dislocated quasi-limit in a dislocated quasi D-metric space are unique.

Definition 2.5 A dislocated quasi D-metric space (X,D) is called complete if every Cauchy's sequence In it is a dislocated quasi D-convergent.

Example 2.6 [5]

Let $X = \{1/2^n : n \in N\}$ Define $D: X x X x X \to [0,\infty)$ as follows

$$D(x,y,z) = \begin{cases} 0 \text{ if } x = y = z \\ \min\{\max\{x,y\}, \max\{y,x\}, \max\{z,x\}\} \text{ otherwise} \end{cases}$$

Define $T:X \to X$ as $Tx = x/2$ for all $x \in X$.

Then X is a complete bounded dislocated quasi D-metric space

Definition: Let (X, D_1) and (Y, D_2) be dislocated quasi D-metric spaces and let $f: X \to Y$ be a function. Then f is continuous to $x_0 \in X$, if for each sequence $\{x_n\}$ which is d_1-q D-convergent to x_0 the sequence $\{f(x_n)\}$ d_2 -q D-convergent to $f(x_0)$ in Y.

Definition 2.8 Let (X,D) be a dislocated quasi D-metric space. A mapping $T:X \to X$ is called contraction if $\exists\ 0 \leq \lambda < 1$ such that $D(Tx,Ty,Tz) \leq \lambda D(x,y,z)$ $\forall x, y.z \in X$

Proposition 2.9 Every convergent sequence in a dislocated quasi D-metric space is 'bi' Cauchy. converse of proposition may not be true.

Lemma 2.10 Let (X,D) be a dislocated quasi D-metric space. If $F:X \to X$ is a contraction function, then $\{f^n(x_0)\}$ is a D-Cauchy's sequence for each $x_0 \in X$

3 Main Results:

Theorem 3.1 Let (X,D) be a complete dislocated quasi D-metric space and let $T: X \to X$ be a continuous mapping satisfying the follows condition

$$D(Tx,Ty,Tz) \leq \alpha \left[\frac{1+D(x,Tx,z)}{1+D(x,y,z)}\right] D(y,Ty,Tz) + \beta D(x,y,z)$$

$for\ all\ x, y \in X, \alpha > 0, \beta > 0, \alpha + \beta < 1. Then\ T\ has\ unique\ fixed\ point.$

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.

$Let\ x_0 \in X$

$T(x_0) = x_1, (Tx_1) = x_2, \ldots\ldots\ldots\ldots, T(x_n) = x_{n+1}, \ldots\ldots$

Consider

$$D(x_n, x_{n+1}, x_{n+2}) = D(Tx_{n-1}, Tx_n, Tx_{n+1})$$

$$\leq \alpha\left[\frac{1+D(x_{n-1}, Tx_{n-1}, x_{n+1})}{1+D(x_{n=1}, x_n, x_{n+1})}\right] D(x_n, Tx_n, Tx_{n+2}) + \beta D(x_{n=1}, x_n, x_{n+1})$$

$$\leq \alpha\left[\frac{1+D(x_{n-1}, x_n, x_{n+1})}{1+D(x_{n=1}, x_n, x_{n+1})}\right] D(x_n, x_{n+1}, x_{n+2}) + \beta D(x_{n=1}, x_n, x_{n+1})$$

Therefore,

$$D(x_n, x_{n+1}, x_{n+2}) = \frac{\beta}{1-\alpha} D(x_{n-1}, x_n, x_{n+1})$$

$$D(x_n, x_{n+1}, x_{n+2}) = \lambda D(x_{n-1}, x_n, x_{n+1})$$

where $\lambda = \dfrac{\beta}{1-\alpha}$ *with* $0 \leq \lambda < 1$. *Similarly, we will show that*

$$D(x_{n-1}, x_n, x_{n+1}) \leq \lambda D(x_{n-2}, x_{n-1}, x_n)$$

and so $D(x_n, x_{n+1}, x_{n+2}) \leq \lambda^2 D(x_{n-2}, x_{n-1}, x_n)$

In this way we have

$$D(x_n, x_{n+1}, x_{n+2}) \leq \lambda^n D(x_2, x_1, x_0)$$

Since $0 \leq \lambda < 1$, as $\lambda^n \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is dislocated quasi D-sequence in the complete dislocated quasi D-metric space X. Thus $\{x_n\}$ is dislocated quasi D-converges to some t_0. Since T is continuous,

we have $T(t_0) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = t_0$.

Thus $T(t_0) = t_0$. Thus T has a fixed point.

Uniqueness: Let x be a fixed point of T. Then by given condition, we have

$$D(x,x,x) = D(Tx,Tx,Tx)$$

$$\leq \alpha\left[\frac{1+D(x,Tx,x)}{1+D(x,x,x)}\right] D(x,Tx,Tx) + \beta D(x,x,x)$$

$D(x,x,x) = D(Tx,Tx,Tx,) \leq (\alpha+\beta) D(x,x,x)$

Which gives $D(x,x,x) = 0$, since $0 \leq (\alpha+\beta) < 1$ and $D(x,x,x) \geq 0$.

Thus $D(x,x,x) = 0$, if x is fixed point of T.

Let $x, y \in X$ *be fixed point of* T. *That is,* $Tx = x, Ty = y$.

Then by condition $D(x,y,z) = D(Tx,Ty,Tz) \leq \beta D(x,y,z)$
Which gives $D(x,y,z) = 0$, since $0 \leq \beta < 1$ and $D(x,y,z) \geq 0$. Similarly $D(y,x,z) = 0$, and hence $x = y = z$.
Thus T has unique fixed point.

Theorem 3.2 Let (X,d) be a complete dislocated D-metric space.
Let $S,T : X \to X$ be D-continuous mapping satisfying:

$$D(Sx,Ty,z) \leq h \max\{D(x,y,z), D(x,Sx,z), D(y,Ty,z)\}$$

for all $x,y \in X$ and $0 < h < 1$. Then S and T have common fixed point.

Proof: Let $x_0 \in X$. Define the sequence x_n by
$X_1=S(x_0)$, $x_2=T(x_1)$,...............$x_n=S(x_{n-1})$, $x_{n+1}=T(x_n)$,.........
Consider
$D(x_n, x_{n+1}, x_{n+1}) = D(Sx_{n-1}, Tx_n, x_{n+1})$

$\leq h \max\{D(x_{n-1}, x_n, x_{n+1}), D(x_{n-1}, x_n, x_{n+1}), D(x_n, x_{n+1}, x_{n+1})\}$

$\leq h\{D(x_{n-1}, x_n, x_{n+1})\}$

Similarly
$D(x_{n-1}, x_n, x_{n+1}) \leq h\{D(x_{n-2}, x_{n-1}, x_n)\}$
and so $D(x_n, x_{n+1}, x_{n+1}) \leq h^2\{D(x_{n-2}, x_{n-1}, x_n)\}$
In this way we have
$D(x_n, x_{n+1}, x_{n+1}) \leq h^n\{D(x_0, x_1, x_2)\}$
Since $o < h < 1$, as $h^n \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is a cauchy sequence in a Complete dislocated D-metric space X. There exists a point $u \in X$ such that $x_n \to u$.
Therefore the subsequences $\{Sx_{n-1}\} \to u$ and $\{Tx_n\} \to u$. Since S and T are continuous functions. So we have $Su = u$ and $Tu = u$.
Uniqueness of common fixed point: Let u,v be a common fixed point of S and G. Then

$D(u,v,v) = D(Su, Tv, v)$

$\leq h \ \max\{D(u,v,v), D(u,u,v), D(v,v,v)\}$

Replacing v by u, we get $D(u,u,u) \leq h\, D(u,u,u)$. Since $0 < h < 1$, we have $D(u,u,u) = 0$. Similarly we have $D(v,v,v) = 0$ and so $u = v$.
Hence the proof is completed.

Theorem 3.3 Let (X, D) be a complete dislocated D-metric space. Let A, B, S, T: $X \to X$ be D-continuous mappings satisfying:
$D(Sx, Ty, z) \leq h \max\{D(Sx, Ty, z), D(Sx, Ax, z), D(Ty, By, z)\}$
for all $x, y \in X$ and $0 < h < 1$. Then A, B, S, T have common fixed point.

Proof: Suppose x_0 is an arbitrary point of X. Define the sequence $\{y_n\}$ by
$Y_{2n} = Ax_{2n} = Tx_{2n+1}$
$Y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$
Consider
$D(y_{2n}, y_{2n+1}, y_{2n+1}) = D(Ax_{2n}, Bx_{2n+1}, y_{2n+1})$
$\leq h \ \max\{ D(Sx_{2n}, Tx_{2n+1}, y_{2n+1}), \ D(Sx_{2n}, Ax_{2n}, y_{2n+1}), \ D(Tx_{2n+1}, Bx_{2n+1}, y_{2n+1}) \}$
$= h \max\{ D(y_{2n-1}, y_{2n}, y_{2n+1}), D(y_{2n-1}, y_{2n}, y_{2n+1}), D(y_{2n}, y_{2n+1}, y_{2n+1}) \}$
$\leq h\, D(y_{2n-1}, y_{2n}, y_{2n+1})$
Similarly
$D(y_{2n-1}, y_{2n}, y_{2n+1}) \leq h\, D(y_{2n-2}, y_{2n-1}, y_{2n})$
and so $D(y_{2n}, y_{2n+1}, y_{2n+1}) \leq h^2\, D(y_{2n-2}, y_{2n-1}, y_{2n})$
In this way we have
$D(y_{2n}, y_{2n+1}, y_{2n+1}) \leq h^n\, (y_2, y_1, y_0)$
Since $0 < h < 1$, as $h^n \to \infty$. Thus $\{y_n\}$ is Cauchy sequence in a complete dislocated D-metric space X.
There exists a point $u \in X$ such that $\{y_n\} \to u$.
Therefore the subsequences $\{ Ax_{2n} \} \to u$, $\{ Bx_{2n+1} \} \to u$, $\{ Sx_{2n+2} \} \to u$ and $\{ Tx_{2n+1} \} \to u$.

So we have Au = u, Bu = u, Su = u, and Tu = u.

Uniqueness of common fixed point :

Let u and v be a common fixed point of A, B, S, T. Then

$$D(u,u,v) \le h \max \{D(Su,Tu,v), D(Su,Au,v), D(Tu,Bu,v)\}$$
$$\le h\, D(u,u,v)$$

Replacing v by u, we get $D(u,u,u) \le hD(u,u,u)$.

Since $0 < h < 1$, we have $D(u,u,u) = 0$. Similarly we have $D(v,v,v) = 0$ and so $u = v$.

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