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Flow Past Streamline Shaped Bodies

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Abstract : In this paper, the fluid motions in which the vortex lines coincides with the stream lines past stream line shaped bodies has been discussed and their aerodynamical application as well as solutions in particular cases are given.

Key words : Stream lines, Vortex lines, Gradual damping.

Introduction :

The paper "Gradual Damping of Solitary Waves" by Garbis H. Kenlegan, [6], suggests that fluid motions of which the vortex lines coincides with steam lines, and which according to Durand [1] represent the state of flow past an aeroplane wing should be capable of representing the graduate damping of wave motion when a ship moves through water. Ramballabh had come across these motions in their study of 'superposable fluid motions' published in the proceedings of Banaras mathematical society vol. II (1940).

One of the important characteristics of these motions is that Bernoullis equation

$$\frac{P}{\rho} + \frac{1}{2} q^2 + V = \text{Constant},$$

where p = fluid pressure, ρ = fluid density

q = fluid velocity, & V = force potential,

is applicable to them although they may represent an unsteady state of flow of a viscous fluid. Two such fluid motions are capable of combining in a natural order.

The results represent exact solutions of the equations of viscous motion and are capable of aerodynamic application as well.

Derivation

The equation of motion [2] of a viscous homogeneous in compressible fluid can be written as

$$\frac{\partial u}{\partial t}-(v\zeta - w\eta) = -\frac{\partial \chi}{\partial x} + \nu\,\nabla^2 u,$$

$$\frac{\partial v}{\partial t}-(w\xi - u\zeta) = -\frac{\partial \chi}{\partial y} + \nu\,\nabla^2 v,$$

and

$$\frac{\partial w}{\partial t}-(u\eta - v\xi) = -\frac{\partial \chi}{\partial z} + \nu\,\nabla^2 w,$$

where ξ, η, ζ are the velocity components, ν the kinematic viscosity,

$$\chi = \frac{P}{\rho} + \frac{1}{2} q^2 + V$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

From these it is apparent that if $\xi = \lambda u$, $\eta = \lambda v$ $\xi = \lambda w$ i.e., if the vortex lines of the motion coincide with the stream lines, the equations of motion reduces to the simple form

(1) $$\frac{\partial u}{\partial t} - \nu\,\nabla^2 u = -\frac{\partial \chi}{\partial x}$$

(2) $$\frac{\partial v}{\partial t} - \nu\,\nabla^2 v = -\frac{\partial \chi}{\partial y}$$

and

(3) $$\frac{\partial w}{\partial t} - \nu\,\nabla^2 w = -\frac{\partial \chi}{\partial t}$$

The necessary and sufficient conditions that these be integrable are

$$\frac{\partial^2 \chi}{\partial z\,\partial y} = \frac{\partial^2 \chi}{\partial y\,\partial z},\ \frac{\partial^2 \chi}{\partial x\,\partial z} = \frac{\partial^2 \chi}{\partial z\,\partial x},\ \frac{\partial^2 \chi}{\partial y\,\partial x} = \frac{\partial^2 \chi}{\partial x\,\partial y}$$

From (1), (2) and (3) we then obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=\nu\,\nabla^2\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right).$$

(4) $$\frac{\partial \xi}{\partial t}=\nu\,\nabla^2\,\xi$$

with two similar equations, namely

(5) $$\frac{\partial \eta}{\partial t}=\nu\,\nabla^2\,\eta$$

and

(6) $$\frac{\partial \zeta}{\partial t}=\nu\,\nabla^2\,\zeta$$

which shows that the vorticity components for a motion of the type under contemplation obey the law of heat conduction through an isotropic medium.

Now, substituting

$$\xi=\lambda u,$$

$$\eta=\lambda v,$$

$$\zeta=\lambda w,$$

in (4), (5) and (6) and assuming λ to be independent of the space variables x, y and z, we get

(7) $$\frac{\partial}{\partial t}(\lambda u)=\nu\,\lambda\nabla^2\,u$$

(8) $$\frac{\partial}{\partial t}(\lambda v)=\nu\,\lambda\nabla^2\,v$$

and

(9) $$\frac{\partial}{\partial t}(\lambda w)=\nu\,\lambda\nabla^2\,w$$

But $$\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}$$

So that $$\lambda u=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}$$

$$=\frac{\partial}{\partial y}\left(\frac{\zeta}{\lambda}\right)-\frac{\partial}{\partial z}\left(\frac{\eta}{\lambda}\right)$$

i.e.,
$$\lambda^2 u = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2}$$

(10)
$$= -\nabla^2 u$$

because from the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

we have $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}$

It can similarly be seen that

(11)
$$\lambda^2 v = -\nabla^2 v$$

and

(12)
$$\lambda^2 w = -\nabla^2 w$$

substituting for $\nabla^2(u, v, w)$ in (7), (8), (9) we have

$$\frac{\partial}{\partial t}(\lambda u) = -\nu \lambda^3 u$$

$$\frac{\partial}{\partial t}(\lambda v) = -\nu \lambda^3 v$$

and
$$\frac{\partial}{\partial t}(\lambda w) = -\nu \lambda^3 w$$

Integrating,

(13)
$$\left.\begin{aligned} \lambda u &= F e^{-\int \nu \lambda^2 dt} \\ \lambda v &= G e^{-\int \nu \lambda^2 dt} \\ \lambda w &= H e^{-\int \nu \lambda^2 dt} \end{aligned}\right]$$

where, F, G, H are functions of x, y, z only, satisfying the equation of continuity

$$\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=0$$

and the equations

(14)
$$\left.\begin{aligned}\frac{\partial H}{\partial y}-\frac{\partial G}{\partial z}&=\lambda F\\ \frac{\partial F}{\partial z}-\frac{\partial H}{\partial x}&=\lambda G\\ \frac{\partial G}{\partial x}-\frac{\partial F}{.\partial y}&=\lambda H\end{aligned}\right]$$

obtained from $\xi' = \lambda u$ etc.

But λ must be an absolute constant. Solution (13) therefore simplifies to

(15)
$$\left.\begin{aligned}\lambda u &= Fe^{-\nu\lambda^2 t}\\ \lambda v &= Ge^{-\nu\lambda^2 t}\\ \lambda w &= He^{-\nu\lambda^2 t}\end{aligned}\right]$$

on substituting these values of u, v and w in the equations of motion (1), (2) and (3), we get

$$\frac{\partial \chi}{\partial x}=\frac{\partial \chi}{\partial y}=\frac{\partial \chi}{\partial t}=0.$$

i.e. χ is a constant, the value of which is the same for all points of the fluid. $\chi=\frac{P}{\rho}+\frac{1}{2}q^2+V$, we conclude that Bernoulli's theorem for steady irrotational flow of a non-viscous homogeneous incompressible fluid is applicable to unsteady viscous flows of the type (15) which possess the following properties

(i) The vortex-lines coincide with stream lines

(ii) The velocity decays exponentially with time, the decay being rapid for fluids of high kinematic viscosity and slow for fluids of low kinematic viscosity. In the case of glycerine [3] (ν = 6.9 at 20° C) or cylinder oil [3] (ν = 10.4 at 20° C) the decay would be rapid. In the case of water [3] (ν = . 01 at 20° C), the decay would be gradual unless the vortcity bears a high ratio to velocity.

Aerodynamical Application

Attention to flows of the above type has been drawn by Durand [4] in the study of flow past an aerofoil where he says 'steady motion is possible only when the vertex lines coincide with the line of flow'.

If the viscosity effect be neglected and the motion be steady, we again have from (1), (2) and (3)

$$\frac{P}{\rho} + \frac{1}{2} q^2 + V = \text{Constant}$$

at all points of the fluid and the value of the constant will be the same for all stream lines, although the flow will be rotational.

In aerodynamical problems the variations in V are small so that Bernoullis equation can be written as

$$\frac{P}{\rho} + \frac{1}{2} q^2 = \frac{p_0}{\rho}$$

where p_0 is the rest pressure.

A Particular Case

To study a particular case, let us suppose $w = 0$
Then from (14),

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0,$$

$$-\frac{\partial G}{\partial z} = \lambda F,$$

$$\frac{\partial F}{\partial z} = \lambda G,$$

$$\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} = 0,$$

the complete solution of which is

$$F = \chi_x \, Cos \, \lambda z + \chi_y \, Sin \, \lambda z$$

$$G = \chi_y \, Cos \, \lambda z - \chi_x \, Sin \, \lambda z \, ,$$

where χ is a function of x and y having continuous derivatives and satisfying the equation

$$\chi_{xx} + \chi_{yy} = 0$$

so that from (15),

$$\lambda u = e^{-\nu \lambda^2 t} \, (\chi_x \, Cos \, \lambda z + \chi_y \, Sin \, \lambda z)$$

$$\lambda v = e^{-\nu \lambda^2 t} \, (\chi_y \, Cos \, \lambda z - \chi_x \, Sin \, \lambda z)$$

$$w = 0,$$

this represents a damped periodic motion arising out of the movement of a streamline shaped body through a viscous fluid. The damping in the case of water will be gradual unless the velocity is large compared with velocity as remarked earlier.

A Special Feature

Fluid motions of which the vertex lines coincide with streamlines have another characteristic property. If u_r, v_r, $r = 1, 2, 3$ be the velocity components of two such motions in the same fluid, and p_1, p_2 the corresponding pressures, we have from (1), (2) and (3)

$$\frac{\partial u_r}{\partial t} - \nu \nabla^2 u_r = -\frac{\partial \chi_1}{\partial x_r}$$

and

$$\frac{\partial v_r}{\partial t} - \nu \nabla^2 v_r = -\frac{\partial \chi_2}{\partial x_r},$$

where χ_1, χ_2 are the values of χ for the two motions and x_1, x_2, x_3 stand for x, y, z representively.

Adding,

$$\frac{\partial}{\partial t}(u_r + v_r) - \nu \nabla^2 (u_r + v_r) = -\frac{\partial}{\partial x_r}(\chi_1 + \chi_2)$$

so that it is possible to super impose one such motion upon another in a way that the velocity of the resulting flow is the vector sum of the velocities of separate flows.

The pressure P for the combined flow will be given by

$$\frac{p}{\zeta} + \frac{1}{2}\sum_{r=1}^{3}(u_r + v_r)^2 + V$$

$$= \frac{p_1}{\zeta} + \frac{1}{2}\sum_{r=1}^{3} u_r^2 + V + \frac{p_2}{\zeta} + \frac{1}{2}\sum_{r=1}^{3} v_r^2 + V,$$

i.e.,

$$p = p_1 + p_2 + \rho\left\{V - \sum_{r=1}^{3} u_r v_r\right\}$$

The pressure head for the combined flow will be in excess of the sum of the pressure heads for separate motions if the force potential $V > \sum_{r=1}^{3} u_r v_r$,which condition may be assumed to be true for slow motion under gravity.

Since the fluid motions are capable of combining in such a simple way, it is expected that an increase in velocity at any point in the fluid will not create any discontinuity in fluid flow.

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