The Nepali Math. Sc. Report
Vol. 29, No. 1 & 2, 2009

Generalized Fixed Point Theorem in Fuzzy Metric Space

KANHAIYA JHA
Department of Mathematical Sciences, School of Science,
Kathmandu University, P. O. Box No. 6250, Kathmandu, Nepal.
E-mail: jhaknh@yahoo.co.in; jhakan@ku.edu.np

Abstract: The main objective of the present paper is to establish a common fixed point theorem for pair of self fuzzy mappings in a fuzzy metric space which generalizes and improves various known results.

AMS Subject Classification: 47 H 10.

Key Words: Fuzzy metric space, Compatible mappings, R-weakly commuting mappings, Reciprocal continuity.

Key Words: Fourier transform, Hilbert Schmidt norm, kernel function.

1. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [14] in 1965. After that, a lot of works have been done regarding fuzzy sets and applications. Deng[3], Erceg [4], Kalva and Seikkala [7] introduced the concepts of fuzzy metric spaces in different ways. In 1975, Kramosil and Michalek [8] introduced the fuzzy metric space by generalizing the concept of probabilistic metric space to fuzzy situation. Grabiec [6] proved the contraction principle in the setting of the fuzzy metric space introduced by Kramosil and Michalek [8]. Grabiec's result was further generalized by Subrahmanyam [12] for a pair of commuting mappings. Since then, a substantial literature has been developed on this topic. Also, George and Veermani [5] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [13] introduced the concept of R-weak commutativity of mappings in fuzzy metric space and Pan[9] introduced the notion of reciprocal continuity of mappings in metric space and proved some common fixed point theorems. Balasubramaniam et. al. [1] proved a fixed point theorem, which generalizes a result of Pant [9] for fuzzy mappings in fuzzy metric space.

Pant and Jha [10] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam et. al. [1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space. Recently, Chugh and Kumar [2] proved a common fixed point theorem for four mappings in fuzzy metric space generalizing the result of Vasuki [13]. The present paper is aimed to prove a fixed point theorem assuming the reciprocal continuity of fuzzy mappings in fuzzy metric space that generalizes the results of Chugh and Kuamr [2], Vasuki [13] and improves various other similar results of fixed points. We also give an example to illustrate our main theorem.

We have used the following notions:

Definition 1.1 ([13]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 1.2 ([11]) A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t–norms if, ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].

Examples of t-norms are $a * b = ab$, $a * b = \min \{a, b\}$.

Definition 1.3 ([8]) The triplet (X, M, *) is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, * is a continuous t–norm and M is a fuzzy et on $X^2 \times [0, \infty)$ satisfying he following conditions: for all x, y, z in X, s, t > 0,

(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x, y, t) = M(y, x, t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$ and $s, t > 0$,

(vi) $\lim_{t\to\infty} M(x, y, t) = 1$, for all $x, y \in X$.

Definition 1.4 ([6]) A sequence $\{x_n\}$ is a fuzzy metric space (X, M, *) is called Cauchy sequence if, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and for each $p > 0$. A fuzzy metric space (X, M, *) is complete if, every Cauchy sequence in X converges in X.

Definition 1.5 ([6]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is said to be convergent to x in X if, $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

Definition 1.6 ([9]) Two self mappings A and S of a metric space (X, d) are called compatible if, $\lim_{n\to\infty} d(Asx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

Definition 1.7 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called compatible if, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some p in X.

Definition 1.8 ([9]) Two self mappings A and S of a metric space (X, d) are called R-weekly commuting at a point x in X if, $d(ASx, SAx) \leq Rd(Ax, Sx)$, for $R > 0$.

Definition 1.9 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called weekly commutating if, $M(ASx, SAx, t) \geq A(Ax, Sx, t)$ for each $x \in X$ and $t > 0$.

Definition 1.10 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called R-weekly commuting provided there exists some real number R such that $M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$ form some $x \in X$ and $t > 0$.

Definition 1.11 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are called pointwise R-weakly commuting on X if, given x in (X, M, *), there exists $R > 0$ such that $M(ASx, SAx, t) \geq M(Ax, Sx, t/R)$.

It is noted that R-weakly commutativity in fuzzy metric space implies weak commutativity only when $R \leq 1$ (Chugh and Kumar [2]).

Definition 1.12 ([1]) Two self mappings A and S of a fuzzy metric space (X, M, *) are said to be reciprocally continuous if, $\lim_{n\to\infty} AS_n = Ap$ and $\lim_{n\to\infty} SAx_n = Sp$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Sx_n = p$ and $\lim_{n\to\infty} Ax_n = p$ for some p in X.

Note that in the metric setting if A and S are both continuous then they are obviously reciprocal continuous. But the converse need not be true (Pant [9]).

2. MAIN RESULTS

Theorem 2.1 Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that

(i) $AX \subseteq TX$, $BX \subseteq SX$,

(ii) $M(Ax, By, t) \geq r(M(Sx, Ty, t))$,

for all x, y ∈ X, where $r : [0, 1] \to [0, 1]$ is continuous function such that $r(t) > t$ for each $0 < t < 1$. If the pair (A, S) or (B, T) is compatible pair of reciprocally continuous mappings, then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. We define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \text{ for } n = 0, 1, 2, 3, ... \quad (1)$$

This can be done by virtue of (i). Then, using (ii), we get

$$\begin{aligned} M(y_{2n}, y_{2n+1}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq r(M(Sx_{2n}, Tx_{2n+1}, t) = r(M(y_{2n-1}, y_{2n}, t)) \\ &> M(y_{2n-1}, y_{2n}, t), \end{aligned}$$

since $r(t) > t$ for $0 < t < 1$. Similarly, we have $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$. So, $\{M(y_{2n}, y_{2n+1}, t)\}$, for $n \geq 0$, is an increasing sequence of positive real numbers in [0, 1] and therefore, tends to a limit $\alpha \leq 1$. We claim that $\alpha = 1$. For this, if $\alpha < 1$, then on letting $n \to \infty$ in relation (2), we get $\alpha \geq r(\alpha) > \alpha$, a contradiction. Hence, we get $\alpha = 1$. Thus, for every $n \in N$,

$$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, t) \text{ and } M(y_n, y_{n+1}, t) \to 1 \text{ as } n \to \infty, \text{ for } t > 0. \quad (3)$$

Now, for any positive integer p, we get

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * ... * M(y_{n+p-1}, y_{n+p}, t/p) \\ &\geq M(y_n, y_{n+1}, t/p) * M(y_n, y_{n+1}, t/p) * ... * M(y_n, y_{n+1}, t/p) \\ &\geq 1 * 1 * ... * 1, \text{ using (3).} \end{aligned}$$

This implies that $M(y_n, y_{n+p}, t) \to 1$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z in X such that $y_n \to z$ as $n \to \infty$. Moreover, we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Suppose A and B are compatible and reciprocally continuous mappings, then by definition, we have $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$. Also, compatibility of A and S yields that $\lim_{n\to\infty} M(ASx_{2n}, SAx_{2n}, t) = 1$, that is, $M(Az, Sz, t) = 1$. Hence, we have $Az = Sz$. Since $AX \subset TX$, there exists a point w in X such that $Az = Tw$. So, using (ii), we get $M(Az, Bw, t) \geq r(M(Sz, Tw, t)) = r(M(Az, Tw, t)) = r(1) = 1$, since $r(t) = 1$ for $t = 1$. This implies that $Az = Bw$.

Thus, we have $Sz = Az = Tw = Bw$.

Again, the pointwise R-weakly commutativity of A and S implies that there exists $R > 0$ such that $M(ASx, SAz, t) \geq M(Az, Sz, t/R) = 1$. That is, $ASz = SAz$ and $AAz = ASz = SSz$. Similarly, the pointwise R-weakly commutativity of B and T implies that $BBW = BTW = TBw = TTw$. So that, using (ii), we have

$$M(Az, AAz, t) = M(Bw, AAz, t) \geq r\,(M(SAz, Tw, t)) > M(AAZ, Az, t).$$

That is, $M(Az, AAz, t) = 1$. Hence, we have $Az = AAz$ and $Az = AAz = SAz$. This implies that Az is a common fixed point of A and S. Similarly, by using (ii), we can show that Bw(= Az) is a common fixed point of B and T. The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (ii). In fact, if u' be another fixed point for mappings A, B, S and T, then, we have $M(u, u', t) = M(Au, Bu', t) \geq r(M(Su, Tu', t)) = r(M(u, u', t)) > M(u, u', t)$, for $r(t) > t$ and hence, we get $u = u'$.

This completely establishes the theorem.

We now give an example to illustrate the above Theorem 2.1.

Example: Let $X = [2, 20]$ and M be the usual fuzzy metric on (X, M, *). Define mappings A, B, S and $T : X \to X$ by

$A2 = 2$, $Ax = 3$ if, $x > 2$;

$Bx = 2$ if, $x = 2$ or > 5, $Bx = 6$ if, $2 < x \leq 5$;

$S2 = 2$, $Sx = 6$ if, $x > 2$;

$T2 = 2$, $Tx = 12$ if, $2 < x \leq 5$, $Tx = x - 5$ if, $x > 5$.

Also, we define $M(Ax, By, t) = \frac{t}{[t + d(x, y)]}$, for all x, y in X and for all $t > 0$. Then, A, B, S and T satisfy all the conditions of the above theorem with $r : [0, 1] \to [0, 1]$ by $r(t) = t^{1/2}$ for $0 < t < 1$ and $r(t) = 1$ for $t = 1$. So that, we have $r(t) > t$ for $0 < t < 1$. Also, $M(Ax, By, t) \geq r(M(Sx, Ty, t)$ for all x, y in X. Moreover, the pair (A, S) and (B, T) are R-weakly commuting and reciprocally continuous mappings on X. Thus, all the conditions of the above Theorem 2.1 are satisfied and $x = 2$ is a common fixed point of A, B, S and T.

Remarks: As Pant [9] has shown that the reciprocally continuous maps need not be continuous. so this result generalizes the results of Chugh and Kumar [2] and

Vasuki [13]. It also improves the results of Balasubramaniam et. al [1], Pant and Jha [10] and other similar results for fixed points.

REFERENCES

1. P. Balasubramaniam, S. Muralishankar and R. P. Pant, Common fixed points of four mappings in a fuzzy metric space, J. Fuzzy Math., 10(2) (2002), 379 – 384.
2. R. Chugh and S. Kumar, Common fixed point theorem in fuzzy metric spaces, Bull. Cal. Math. Soc., 94(1) (2002), 17 – 22.
3. Z. K. Deng, Fuzzy pseudo-metric spaces, J. Math. Anal. Appl., 86(1982), 74 – 95.
4. M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. Anal. Appl. 69 (1979), 202 – 230.
5. A. George and P. Veeramani, On some results in fuzzy metric space, Fuzzy Sets and Systems, 64(1994), 395 – 399.
6. M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27(1988), 385 – 389.
7. O. Kalva and S. Seikkala, On fuzzy metric space, Fuzzy sets and Systems, 12(3) (1984), 215 – 229.
8. O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika, 11(1975), 326 – 334.
9. R. P. Pant, Common fixed points of four mappings, Bull. Cal. Math. Soc., 90(1998), 281 – 286.
10. R. P. Pant and K. Jha, A remark on common fixed points of four mappings in a fuzzy metric space, J. Fuzzy Math., 12(2) (2004), 433 – 437.
11. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math., 10 (1960), 314 – 334.
12. P. V. Subarahmanyam, A common fixed point theorem in fuzzy metric space, Inform Sci., 83 (1995), 103 – 112.
13. R. Vasuki, Common fixed points for R-weakly commuting mappings in fuzzy metric spaces, Indian J. Pure Appl. Math., 30 (1999), 419 – 423.
14. L. A. Zadeh, Fuzzy sets, Inform and Control, 89(1965), 338 – 353.