The Nepali Math. Sci. Report
Vol. 24. No.1. 2005.

Hardy's Uncertainty Principle on $\mathbb{R}^+=[0,\infty)$

CHET RAJ BHATTA

Abstract: Hardy's uncertainty principle states that if the function f is "very rapidly decreasing" then the Fourier transform can not also be "very rapidly decreasing" unless f is identically zero. In this paper we discuss some variants of Hardy's theorem on $\mathbb{R}^+=[0, \infty)$.

Keywords: Uncertainty principle, Fourier transform pair, Laplace transform, very rapidly decreasing.

1. Introduction

It is well-known simple fact that if a function f on $\mathbb{R}$ is compactly supported, then its Fourier transform $\hat{f}$ can not also be compactly supported, unless $f = 0$. More generally, we have the following principle in classical Fourier analysis: If the function f is "very rapidly decreasing" then the Fourier transform can not also be "very rapidly decreasing" unless f is identically zero. An important result making this precise is the following theorem. There are several ways of measuring "Concentration". One way of measuring concentration is by considering the decay of the function at infinity and another natural way of measuring 'concentration' is in terms of the supports of the function f and its Fourier transform $\hat{f}$.

Hardy Theorem 1.1 [1] : *Let α, β and C be positive real numbers and suppose that f is measurable function on $\mathbb{R}$ such that*

(i) $|f(x)| \le C \exp(-\alpha\pi x^2)$ for all $x \in \mathbb{R}$

(ii) $|\hat{f}(\xi)| \le C \exp(-\beta\pi\xi^2)$ for all $\xi \in \mathbb{R}$

If $\alpha\beta > 1$ then $f = 0$ almost everywhere. If $\alpha\beta < 1$ then there are infinitely many linearly independent functions satisfying (i) and (ii) and if $\alpha\beta = 1$, then

$$f(x) = C \exp(-\alpha\pi x^2) \text{ for some constant } C.$$

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) \exp(-2\pi ixy)\, dx,\ y \in \mathbb{R}$$

Definition 1.2: A function f is said to be "exponential type" if $|f(\gamma)| \le \text{constant} \times e^{T|\gamma|}$ for some $T < \infty$.

Definition 1.3: For a measurable function f on $\mathbb{R}^+ = [0, \infty)$ the Laplace transform $\mathcal{L}f$ of f is defined by

$$\mathcal{L}f(\gamma) = \int_0^\infty f(t)\exp(-\pi t\gamma)\,dt$$

If Re $\gamma > 0$ and $f \in L^1(\mathbb{R}^+)$ then $\mathcal{L}f$ is well-defined. If in addition f satisfies $|f(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}^+$ then $\mathcal{L}f(\gamma)$ is defined for all $\gamma \in \mathbb{C}$ and is holomorphic function on $\mathbb{C}$.

The following is a simple deduction from Hardy's theorem on $\mathbb{R}^+$.

Theorem 1.3: *Let f be a measurable function on $\mathbb{R}^+$ satisfying*

(i) $|f(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}^+$

(ii) $|\mathcal{L}f(\gamma)| \le C\exp(-\beta(\mathrm{Im}\gamma)^2)$ for all $\gamma \in i\mathbb{R}$

If $\alpha\beta > ¼$ then $f = 0$ a.e.

Proof: Extend f to $\tilde{f}$ on $\mathbb{R}$ by defining O on $\{x : x < 0\}$ then $\tilde{f}$ is a measurable functions on $\mathbb{R}$ and satisfy $|\tilde{f}(x)| \le C\exp(-\alpha x^2)$ for all $x \in \mathbb{R}$

$$\text{For } \gamma \in \mathbb{R},\ \left|\hat{\tilde{f}}(\gamma)\right| = \left|\int_{-\infty}^{\infty} \tilde{f}(x)\exp(-2\pi i x\gamma)\,dx\right|$$

$$= \left|\int_0^\infty f(x)\exp(-2\pi i x\gamma)\,dx\right|$$

$$= |\mathcal{L}f(2i\gamma)|$$

$$\le C\exp(-4\beta\gamma^2)$$

Since $4\alpha\beta > 1$ so by Hardy's theorem on $\mathbb{R}$, $\tilde{f} = 0$ a.e. Thus $f = 0$ a.e.

Theorem 1.4: *Let f be a measurable function on $\mathbb{R}^+$ satisfying*

(i) $|f(x)| \le C\exp(-\alpha\pi x^2)$ for all $x \in \mathbb{R}^+$

(ii) $|\mathcal{L}f(\gamma)| \le C\exp(-\beta\pi\gamma^2)$ for all $\gamma \in \mathbb{R}$

If $\alpha\beta > 1$ then $f = 0$ a.e.

Proof: Suppose that $\alpha = \beta = 1$ and $\mathcal{L}f(\gamma)$ is a even function i.e. $\mathcal{L}f(\gamma) = \sum C_n \gamma^{2n}$. Since the function $u(\gamma) = \gamma^k$, $k \in \mathbb{R}$ is a holomorphic function in the cut plane $\{\gamma = R\exp(i\theta)\ ;\ R > 0, |\theta| < \pi\}$ where $\gamma^k = R^k\exp(ik\theta)$, we can define a function

$$h(\gamma) = \mathcal{L}f(\sqrt{\gamma}) = \sum C_n \gamma^n.$$

$$|h(\gamma)| = \left|\int_0^\infty f(t)\exp\left(-\pi\sqrt{\gamma}\,t\right)dt\right|$$

$$\le C\int_0^\infty \exp\left(-\pi t^2\right)\exp\left(-\pi\sqrt{R}\,\mathrm{Cos}\,\tfrac{\theta}{2}\,t\right)dt$$

$$= C\exp\left(\tfrac{\pi}{4}R\,\mathrm{Cos}^2\tfrac{\theta}{2}\right)\int_0^\infty \exp\left(-\pi\left(t+\tfrac{1}{2}\sqrt{R}\,\mathrm{Cos}\tfrac{\theta}{2}\right)^2\right)dt$$

$$\leq C'\exp\left(\tfrac{\pi}{4}\mathrm{Cos}^2\tfrac{\theta}{2}R\right)$$

(1) $$\leq C'\exp\left(\tfrac{\pi}{4}R\right) \text{ for some } C' > 0$$

Which is independent of θ. Thus h is a exponential type

For $0 < \delta < \pi$

$$\left|\exp\left(\frac{\pi i\gamma e^{-i\delta/2}}{\mathrm{Sin}\,\delta/2}\right)h(\gamma)\right| = \exp\left(\frac{-\pi R\sin(\theta-\frac{\delta}{2})}{\mathrm{Sin}\delta/2}\right)|h(\mathrm{Re}^{i\theta})|$$

If $\theta = 0$ then $\mathrm{Re}^{i\theta} = R > 0$ so

$$\left|\exp\left(\frac{\pi i\gamma e^{-i\delta/2}}{\mathrm{Sin}\delta/2}\right)h(\gamma)\right| = \exp(\pi R)\,|h(R)|$$

$$\leq \exp(\pi R)\,|\mathcal{K}f(\sqrt{R})|$$

$$\leq C\exp(\pi R)\exp(-\pi R) \leq C$$

If $\theta = \delta$ then, $$\left|\left(\exp\frac{i\pi\gamma e^{-i\delta/2}}{\mathrm{Sin}\delta/2}\right)h(\gamma)\right| = \exp(-\pi R)\,|h(\mathrm{Re}^{i\delta})|$$

$$\leq C'\exp(-\pi R)\exp(\pi/4\,R) \text{ using (1)}$$

$$\leq C'$$

Now we apply Phragmen - Lindelöf's theorem to the sector $0 < \theta < \delta$ to get

$$|h(\gamma)| \leq K\exp\left(\frac{\pi R\sin(\theta-\frac{\delta}{2})}{\mathrm{Sin}\delta/2}\right), K = \mathrm{Max}(C, C')$$

Now taking $\delta \uparrow \pi$, we have

$$|h(\gamma)| \leq K\exp(-\pi R\ \mathrm{Cos}\,\theta) \text{ for } 0 \leq \theta \leq \pi$$

$$\Rightarrow |\exp(\pi\gamma)\,h(\gamma)| \leq K, \gamma = \mathrm{Re}^{i\theta},\ 0 \leq \theta \leq \pi$$

A similar argument will hold for the lower half plane so

$$|\exp(\pi\gamma)\,h(\gamma)| \leq K \text{ for } -\pi \leq \theta \leq 0, \gamma = \mathrm{Re}^{i\theta}$$

Therefore $g(\gamma) = \exp(\pi\gamma)\,h(\gamma)$ is bounded and holomorphic in $\mathbb{C}$. Hence by Liouville's theorem there is a constant $M > 0$ such that

$$h(\gamma) = M\exp(-\pi\gamma)$$

Thus $$\mathcal{K}f(\gamma) = M\exp(-\pi\gamma^2).$$

Suppose now that $\mathcal{K}f(\gamma)$ is an odd function i.e.

$$\mathcal{K}f(\gamma) = \sum C_n\gamma^{2n+1},\ \mathcal{K}f(0) = 0. \text{ For } \gamma \neq 0,\ \gamma^{-1}\mathcal{K}f(\gamma) = \sum C_n\gamma^{2n}.$$

Thus by even case $\gamma^{-1}\mathcal{K}f(\gamma) = M\exp(-\pi\gamma^2)$. But for $\gamma \in \mathbb{R}$, we have

$$|\mathscr{F}f(\gamma)| \leq C \exp(-\pi\gamma^2)$$

Therefore, $M|\gamma|\exp(-\pi\gamma^2) \leq C\exp(-\pi\gamma^2)$

i.e. $M|\gamma| \leq C$ for all $\gamma \in \mathbb{R}$ which is possible only if $M = 0$.

Hence $\mathscr{F}f(\gamma) = 0$.

In general, we break $\mathscr{F}f$ into even and odd part i.e.

$$\mathscr{F}f(\gamma) = \tfrac{1}{2}(\mathscr{F}f(\gamma) + \mathscr{F}f(-\gamma)) + \tfrac{1}{2}(\mathscr{F}f(\gamma) - \mathscr{F}f(-\gamma))$$
$$= g_1(\gamma) + g_2(\gamma) \text{ (say)}$$

$g_1(\gamma)$ is an even function satisfying $|g_1(\gamma)| \leq C\exp(-\pi\gamma^2)$ for all $\gamma \in \mathbb{R}$. So the function $h_1(\gamma) = g_1(\sqrt{\gamma})$ is of exponential type as

$$|h_1(\gamma)| \leq \tfrac{1}{2}\left(\mathscr{F}f(\sqrt{\gamma}) + \mathscr{F}f(-\sqrt{\gamma})\right)$$

and

$$\left|\mathscr{F}f(-\sqrt{\gamma})\right| \leq \int_0^\infty |f(t)|\left|\exp(\pi\sqrt{\gamma}\,t)dt\right|$$

$$\leq C\int_0^\infty \exp(-\pi t^2)\exp\left(\pi\sqrt{R}\operatorname{Cos}\tfrac{\theta}{2}t\right)dt$$

$$= C\exp\left(\pi\frac{R}{4}\operatorname{Cos}^2\frac{\theta}{2}\right)\int_0^\infty \exp\left(-\pi\left(t - \frac{\sqrt{R}}{2}\operatorname{Cos}\frac{\theta}{2}\right)^2\right)dt$$

$$\leq C'\exp\left(\frac{\pi R}{4}\right)$$

Hence, $|h_1(\gamma)| \leq C'\exp\left(\frac{\pi R}{4}\right)$

Thus, $g_1(\gamma) = K\exp(-\pi\gamma^2)$ and $g_2(\gamma) = 0$ and so

$$\mathscr{F}f(\gamma) = K\exp(-\pi\gamma^2)$$

If $\alpha = \beta > 1$. Then

$$|f(x)| \leq C\exp(-\alpha x^2) \leq C\exp(-x^2)$$

and $|\mathscr{F}f(\gamma))| \leq C\exp(-\beta\gamma^2) \leq C\exp(-\gamma^2)$

So, $\mathscr{F}f(\gamma)) = K\exp(-\pi\gamma^2)$ by the above case.

For $x \neq 0$.

$$f(x) = M\lim_{a\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\pi(a+ib)^2\right)\exp(ibx)\,db$$

$$= M\lim_{a\downarrow 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\pi\left(a^2 - b^2 + ib\left(2a - \tfrac{x}{\pi}\right)\right)\right)db$$

$$= M\lim_{a\downarrow 0}\exp(-\pi a^2)\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\exp(\pi b^2)\cos b\left(2a - \tfrac{x}{\pi}\right)db\right.$$

$$\left. + i\int_{-\infty}^{\infty}\exp(\pi b^2)\right]\sin b\left(2a - \tfrac{x}{\pi}\right)db$$

Therefore, $\lim_{x \to 0+} f(x) = M \cdot \frac{1}{\pi} \int_0^\infty \exp(\pi b^2)\, db \geq \frac{M}{\pi} X$ for all $X \geq 0$

Hence we must have $M = 0$ and therefore $\mathcal{K} f(\gamma) = 0$ for all γ i.e. f = o a.e

REFERENCES

[1] Conway, J.B. 1973. *Functions of One Complex Variable Springer Ver*lag, New York Inc.

[2] Dym, H. and Mckean, H.P. 1972. *Fourier series and integrals*, Academic Press.

[3] Hardy, G.H. 1933. *A theorem concerning Fourier trans*form. J. London. Math. Soc. 8, 227–237.

[4] Kaniuth, E. and Kumar, A. 2001. *Hardy's theorem for simply connected Liegroups*, Math. Proc. Comb. Phil. Soc. 487–494.

[5] Oberhettinger, F. and Badii, L. *Table of Laplace transforms*, Springer Verlag, Berlin Heidelberg. New York 1995.

[6] Sitaram, M. Sundari and S. Thangavelu. *Uncertainty Principles on Certain Liegroups*, Proc. Indian Acad. Sci. (Math. Sci), Vol 105, No. 2, 135–151.

CHET RAJ BHATTA
Central Department of Mathematics,
Tribhuvan University,
Kirtipur, Kathmandu, Nepal.