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Hypergeometric series involving Ramanujan's mock–theta functions

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Abstract: In this paper, we shall deal with certain aspects of hypergeometric series involving Ramanujan's mock-theta functions. In the concluding sections, certain generating relations of mock-theta functions of order three and five have also been deduced.

Keywords: Bailey's transformations, Generating relations, Hypergeometric series, Mock-theta functions, Saalschutzian series, Theta functions.

1.1 Introduction (A Historical Survey):

S. Ramanujan (1887–1923) was an Indian mathematician whose originality and natural ability for calculation enabled him to develop innovative mathematical concepts enough during the early 20th century. Ramanujan, three months before his death, had written his last letter to G. H. Hardy, a mathematician at the University of Cambridge, England, discovered much interesting functions which I recently call mock-theta functions unlike the false theta functions due to L. J. Rogers, they enter into mathematics as beautifully as the ordinary theta functions." The mock-theta function is the last gift of Ramanujan to the mathematical world.

Due to Hardy [4] a mock-theta function is a function defined by a q-series convergent $|q|<1$ for which we may be able to calculate asymptotic formulae when q tends to a rational point of the unit circle of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformations.

1.2 Notations and definitions:

The generalized hypergeometric series, both ordinary and basic have been a very significant tool in the derivation of the generating relations for mock theta functions. The usual hypergeomtric notation shall be followed throughout, in what follows. As usual, let for any positive integer n,

$$(a)_n = a(a+1)..........(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, n>0, (a)_0 = 1, a \neq 0, (a)_{-n} = \frac{(-1)^n}{(1-a)_n}$$

Then the generalized hypergeometric series is defined by

$$(1) \quad F_1(q) = \sum_{k=0}^{\infty} \frac{q^{2k^2+2k}}{[q;q^2]_{k+1}}$$

The series 1.2(1) converges for all z if $r \leq s$ while it converges for only $z \neq 0$ if $r > s+1$, for $r = s+1$ and also when $z = 1$ provided that $R_e[\sum(b) - \sum(a)] > -1$, When $r = s+1$, the series is called Saalschutzian when $R_e[\sum(b) - \sum(a)] = 1$ and well poised when $1 + a_1 = b_1 + a_2 = \ldots = b_s + a_{s+1}$.

If any one of the numerator parameters in 1.2(1) is zero or a negative integer than ${}_rF_s$ reduces to a polynomial but if any b parameter is a negative integer-N (say) where $N = 1, 2, 3, \ldots$ (unless any of the a parameters is also a negative integer-M say, where $M = N, N+1, N+2, \ldots$, or zero), the ${}_rF_s$ series is not defined.

Let, for $|q^k| < 1$,

$$(a;q^k)_n = (1-a)(1-aq^k) \ldots (1-aq^{k(n-1)}), n > 0,$$
$$(a;q^k)_0 = 1$$

and $(a;q^k)_\infty = \prod_{n=0}^{\infty}(1-aq^{kn})$

Then a basic hypergeometric series is defined by

$$(2) \qquad {}_{r+1}\Phi_r^{q^k}\left[\begin{matrix} a_1, \ldots, a_{r+1}; z \\ b_1, \ldots, b_r \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1;q^k)_n \ldots (a_{r+1};q^k)_n z^n}{(b_1;q^k)_n \ldots (b_r;q^k)_n (q^k;q^k)_n}$$

When $k = 1$ in the above symbols, including that in ${}_{r+1}\Phi_r$, it shall be omitted from the symbols, so $(a;q)_n = (a)_n$ etc. G.N.Watson [7] made use of the basic hypergeometric series to get new definitions of mock theta functions. For this, he used a limiting case of a transformation connecting a terminating well poised ${}_4\Phi_3$ which Watson himself discovered many years back.

We shall often use the abbreviated notation $(a_1, a_2, \ldots a_m; q^k)_n$ to denote $(a_1;q^k)_n (a_2;q^k)_n, \ldots (a_m;q^k)_n$ for all non-negative integers n, where $(a;q^k)_n = \prod_{r=0}^{n-1}(1-aq^{kr})$ and $(a;q^k)_\infty = \prod_{r=0}^{n-1}(1-aq^{kr})$, if $k = 1$ and there is no chance of any confusion, then q^k shall be omitted from the $(a;q^k)_n$ symbol.

So, let $(a)_n=(1-a)(1-aq)\ \dots\ \left(1-aq^{n-1}\right)$, $(a)_0=1$, $\left(a;q^k\right)_n$ $(a)_\infty=(1-aq^n)$, $|q|<1$, we define a generalized basic hypergeometric series by

$$(3)\qquad {}_r\Phi_s\left[\begin{matrix}(a_r);z\\(\beta_s)\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{[(a_r)]_n\,z^n}{(q)_n\,[(\beta_s)]_n},$$

where r is the number of parameters (a_r) and s is the number of parameters (β_s). We shall use the notation ${}_r\Phi_s\left(q^k\right)(z)$ to denote a ${}_r\Phi_s$ series that all the terms are on the base q^k.

If q^k is not written in Φ symbol, the transformation formula is then deduced by

$$(4)\qquad {}_8\Phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},c,d,e,f,g,\dfrac{a^2q^2}{cdefg}\\ \sqrt{a},-\sqrt{a},aq/c,aq/d,aq/e,aq/f,aq/g\end{matrix}\right]$$

$$=\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/fg)(1-aq^n/ge)(1-aq^n/ef)}{(1-aq^n/e)(1-aq^n/f)(1-aq^n/g)(1-aq^n/efg)}\right]\times{}_4\Phi_3\left[\begin{matrix}aq/cd,e,f,g;q\\ efg/a,aq/c,aq/d\end{matrix}\right],$$

provided e, f, or g is of the term q^{-N}, where N is a positive integer. Assuming $a\rightarrow 1$, $e\rightarrow\infty$, $g\rightarrow\infty$ and taking $c=\exp(i\theta)$, $d=\exp(-i\theta)$, 1.2 (4) becomes

$$(5)\qquad 1+\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)2-\cos\theta)q^{n(2n+1)}/2}{1-2q^n\cos\theta+q^{2n}}=\prod_{r=1}^{\infty}(1-q^r)\left[1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-2q^m\cos\theta+q^{2m})}\right]$$

1.2(5) is the most important relation to obtain the new alternative definition of the Ramanujan's mock theta functions. Substituting $\theta=\Pi$, $\theta=\dfrac{\Pi}{2}$, $\theta=\dfrac{\Pi}{3}$ respectively in 1.2(5), we have

$$(6)\qquad f(q)\prod_{r=1}^{\infty}(1-q^r)=1+4\sum_{n=1}^{\infty}\frac{(-)^n q^{n(3n+1)/2}}{1+q^n},$$

$$(7)\qquad \phi(q)\prod_{r=1}^{\infty}(1-q^r)=1+2\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)\,q^{n(3n+1)/2}}{1+q^{2n}}$$

$$(8)\qquad \chi(q)\prod_{r=1}^{\infty}(1-q^r)=1+\sum_{n=1}^{\infty}\frac{(-)^n(1+q^n)q^{n(3n+1)/2}}{1-q^n+q^{2n}}$$

These are the relations providing alternative definitions of $f(q)$, $\phi(q)$ and $\chi(q)$ respectively

1.3 Mock theta functions associated partially:

Let $F(q)=\sum_{n=0}^{\infty}\Psi n(q)$ be a mock theta function. Then the corresponding partial mock theta function is defined by $F_m(q)=\sum_{n=0}^{\infty}\Psi_n(q)$. Thus the partial mock theta functions of order three and five are defined below:

Mock theta function of Partial mock theta function order three:of order three :

(i) $F(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k^2}$ $\quad f_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k^2}$

(ii) $\Phi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q^2;q^2]_k}$ $\quad \Phi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q^2;q^2]_k}$

(iii) $\Psi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$ $\quad \Psi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$

(iv) $\chi(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-wq;-w^2q;q]_k},$ $\quad \chi_n(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-wq;-w^2q;q]_k}$

where w is the cube root of unity.

(v) $w(q)=\sum_{k=0}^{\infty}\frac{e^{2k(k+1)}}{[-q;q^2]_{k+1}^2}$ $\quad w_n(q)=\sum_{k=0}^{\infty}\frac{e^{2k(k+1)}}{[-q;q^2]_{k+1}^2}$

(vi) $u(q)=\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{[-q;q^2]_{k+1}}$ $\quad u_n(q)=\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{[-q;q^2]_{k+1}}$

(vii) $p(q)=\sum_{k=0}^{\infty}\frac{q^{2k(k+1)}}{[wq;w^2q;q^2]_{k+1}}$ $\quad p_n(q)=\sum_{k=0}^{\infty}\frac{q^{2k(k+1)}}{[wq;w^2q;q^2]_{k+1}}$

Mock theta functions Partial mock theta functions of order five : of order five :

(i) $f_0(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$ $\quad f_{0m}(q)=\sum_{k=0}^{\infty}\frac{q^{k^2}}{[-q;q]_k}$

(ii) $F_0(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2}}{[q;q^2]_k}$ $\quad F_{0,n}(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2}}{[q;q^2]_k}$

(iii) $1+2\Psi_0(q)=\sum_{k=0}^{\infty}[-1;q]_k\, q^{\binom{k+1}{2}}$ $\quad 1+2\Psi_{0,n}(q)=\sum_{k=0}^{\infty}[-1;q]_k\, q^{\binom{k+1}{2}}$

(iv) $\Psi_0(q)=\sum_{k=0}^{\infty}\left[-1;q^2\right]_k q^{k^2}$ $\qquad \Psi_{0,n}(q)=\sum_{k=0}^{\infty}\left[-1;q^2\right]_k q^{k^2}$

(v) $f_1(q)=\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{[-q;q]_k}$ $\qquad f_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^{k^2+k}}{[-q;q]_k}$

(vi) $F_1(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2+2k}}{\left[q;q^2\right]_{k+1}}$ $\qquad F_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^{2k^2+2k}}{\left[q;q^2\right]_{k+1}}$

(vii) $\Psi_1(q)=\sum_{k=0}^{\infty}[-q;q]_k\, q^{\binom{k+1}{2}}$ $\qquad \Psi_{1,n}(q)=\sum_{k=0}^{\infty}[-q;q]_k\, q^{\binom{k+1}{2}}$

(viii) $\Phi_1(q)=\sum_{k=0}^{\infty}\left[-q;q^2\right]_k q^{(k+1)^2}$ $\qquad \Phi_{1,n}(q)=\sum_{k=0}^{\infty}\left[-q;q^2\right]_k q^{(k+1)^2}$

(ix) $\chi_0(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_k}$ $\qquad \chi_{0,n}(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_k}$

(x) $\chi_1(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_{k+1}}$ $\qquad \chi_{1,n}(q)=\sum_{k=0}^{\infty}\frac{q^k}{\left[q^{k+1};q\right]_{k+1}}$

1.4 Generating relations for mock theta function of order three:

In 1944, Bailey [3] established the following transformations:

If $\quad \beta_n=\sum_{r=0}^{n}a_r U_{n-r}V_{n+r}$ and $\gamma_n=\sum_{r=0}^{n}\delta_{r+n}u_r v_{r+2n}$

then under suitable convergence conditions:

(1) $$\sum_{n=0}^{\infty}\alpha_n\gamma_n=\sum_{n=0}^{\infty}\beta_n\delta_n\ ,$$

where u_r , v_r , α_r and δ_r are the functions of r only and the series β_n and γ_n shall be convergent. In this section, we shall make use of the above so called Bailey's transformation in order to establish the generating relations for partial mock theta functions of order three defined in 1.3(i)-1.3 (vii). If we take $u_r=v_r=1$ and $\delta_r=z^r$ in 1.4(1), the Bailey's transformation yields

(2) $$(1-z)^{-1}\sum_{n=0}^{\infty}a_n Z^n=\sum_{n=0}^{\infty}\beta_n Z^n \quad \text{whenever} \quad \beta_n=\sum_{r=0}^{n}a_r\ ,\ \gamma_n=\frac{Z^n}{1-z}.$$

We shall now make use of 1.4(2) for the derivation of the mock theta functions of order three and five in the following sections:

1.5. Main generating results involving partial Mock-theta functions of order three:

(i) Taking $\alpha_r = \dfrac{q^{r^2}}{[-q;q]_r^2}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{[-q;q]_r^2} = f_n(q)$$

Putting these values of and in 1.4 (2), we get

$$(1) \quad \sum_{n=0}^{\infty} f_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_r^2} z^n$$

(ii) Taking $\alpha_r = \dfrac{q^{r^2}}{\left[-q^2;q^2\right]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{\left[-q;q^2\right]} = \Phi_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(2) \qquad \sum_{n=0}^{\infty} \Phi_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{\left[-q^2;q^2\right]_n}$$

(iii) Taking $\alpha_r = \dfrac{q^{n^2}}{\left[q;q^2\right]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{\left[q;q^2\right]_r} = \Psi(n)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(3) \qquad \sum_{n=0}^{n} \Psi(n) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{\left[q;q^2\right]_n} z^n$$

(iv) Taking $\alpha_r = \dfrac{q^{r^2}}{\left[-wq;-w^2q;q\right]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r^2}}{\left[-wq;-w^2q;q\right]_r} = \chi_n(q),$$

where w is the cube root of unity.

Putting these values of α_n and β_n in 1.4(2), we get

$$(4) \qquad \sum_{n=0}^{\infty} f_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{r^2}}{\left[-wq;-w^2q;q\right]_n} z$$

(v) Taking $\alpha_r = \dfrac{q^{2r(r+1)}}{\left[q;q^2\right]_{r+1}^2}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{\left[q;q^2\right]_{r+1}} w_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

(5) $$\sum_{n=0}^{\infty} w_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\left[q;q^2\right]_{n+1}^2} w_n(q)$$

(vi) Taking $\alpha_r = \dfrac{q^{r(r+1)}}{\left[-q;q^2\right]_{r+1}}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{r(r+1)}}{\left[-q;q^2\right]_{r+1}} = u_n(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

(6) $$\sum_{n=0}^{\infty} v_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{\left[-q;q^2\right]_{n+1}} z^n$$

(vii) Taking $\alpha_r = \dfrac{q^{2r(r+1)}}{\left[wq;w^2q;q^2\right]_{r+1}}$, we find

$$\beta_n = \sum_{n=0}^{\infty} \frac{q^{2r(r+1)}}{\left[wq;w^2q;q^2\right]_{r+1}},$$

where w is the cube root of unity

Putting these values of α_n and β_n in 1.4(2), we get

(7) $$\sum_{n=0}^{\infty} \rho_n(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{\left[-wq;-w^2q;q\right]_{n+1}} z^n$$

1.6 Main generating results involving partial mock theta functions of order five:

(i) Taking $a_r = \dfrac{q^{r^2}}{[-q;q]_r}$, we find $\beta_n = \sum_{r=0}^{n} \dfrac{q^{r^2}}{[-q;q]_r} = f_{0,n}(q)$.

Putting these values of α_n and β_n in 1.4(2), we get

(1) $$\sum_{n=0}^{\infty} f_{0,n}(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n} z^n$$

(ii) Taking $a_r = \frac{q^{2r^2}}{\left[-q;q^2\right]_r}$, we find

$$\beta_n = \sum_{r=0}^{n} \frac{q^{2r^2}}{[-q;q2]_r} = F_{0,n}(q)$$

Putting these values of α_n and β_n in 1.4(2), we get

$$(2) \qquad \sum_{n=0}^{\infty} F_{0,n}(q) z^n = (1-z)^{-1} \sum_{n=0}^{\infty} = \frac{q^{2n^2}}{\left[-q;q^2\right]_n} z^n$$

Similarly, making use of 1.4(2), we can establish the generating results of mock theta functions 1.3(iii) – 1.3(x) of order five also.

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