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Infinitesimal Variation of Hypersurfaces of an Almost *r*-Contact Hyperbolic Structure Manifold

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Summary: The Infinitesimal variation of the structure tensors of an almost contact metric structure induced on the hyper surface of a Kahlerian manifold under various conditions has been studied by Yano. In this paper we have studied the infinitesimal variation of the structure tensors of an almost r-contact hyperbolic structure induced on the hyper surface of a differentiable manifold equipped with an almost r-contact hyperbolic structure.

1. Introduction :

Let M^{n+r} be an $(n+r)$ dimensional differentiable manifold of differentiability class C^{∞}. Let there exist on M^{n+r} a C^{∞} vector valued linear function F, rC^{∞} linearly independent and non zero contravariant vector fields $T^1, T^2, \ldots T^r$ such that

$$F^2 X = X + \sum_{l=1}^{r} A_l(x) T^l \tag{1.1}$$

for arbitrary vector field X on M^{n+r}. Also

$$F(X) = \overline{X} \tag{1.2}$$

In view of (1.1), let M^{n+r} be endowed with the Riemannian metric G such that it satisfies the following condition.

$$G(\overline{X}, \overline{Y}) + G(X, Y) + \sum_{l=1}^{r} A_l(X) A_l(Y) = 0 \tag{1.3}$$

Thus M^{n+r} satisfying the conditions (1.1) and (1.3) will be called an almost r-contact hyperbolic structure manifold [2].

In M^{n+r} the following results hold

(1.4)
(a) $\overline{T}^{l} = 0$,
(b) $A_l(\overline{X}) = 0$, for arbitrary vector field X.
(C) $A_l(T^m) + \delta_l^m = 0$,

Where δ_l^m is Kronecker delta and ℓ, m take the values $1, 2, \ldots, r$.

Let us imbed a hypersurface M^{n+r-1} into M^{n+r} by the isometric immersion $b : M^{n+r-1} \rightarrow M^{n+r}$. Corresponding to this we have the Jacobian b^* of b denoted by B which carries $T_q(M^{n+r-1})$ into $T_b(q) M^{n+r}$ injectively. Since the immersion is isometric, we have

$$G(BX, BY) \circ b = g(X,Y). \tag{1.5}$$

g being the metric induced on the hyper surface and X,Y denote arbitrary vector fields, We have

$$G(BX, N) = 0, \tag{1.6}$$

$$G(N,N) = 1. \tag{1.7}$$

The transformation equations are

$$FBX = BfX + \alpha(X)N, \tag{1.8}$$

$$FN = BP + \eta N, \tag{1.9}$$

where f is a tensor field of type (1.1) and α is a 1-form on M^{n+r-1}. From equation (1.8) and the relations

(1.10)
(a) $T^l = Bt_l + P_l N$,
(b) $A_l(BX) \circ b = a_l(X)$,
(c) $\alpha(X)P = 0$.

we get

$$f^2 X = X + \sum_{l=1}^{r} a_l(X)\, t^l \tag{1.11}$$

The metric g in (1.5) is found to satisfy

$$g(fX, fY) + g(X,Y) + \sum_{l=1}^{r} a_l(X)\, a_l(Y) = 0. \tag{1.12}$$

Consequently an almost r-contact hyperbolic structure gets induced on M^{n+r-1}.

Let D be the Riemannina connecxion induced on M^{n+r-1}. Then we have the Gauss and Weingartan equations [1].

(1.13) $$E_{BX}\, BY = BD_X Y + H(X,Y)N,$$

(1.14) $$E_{BX}\, N = -B'HX,$$

Where H is the 2[nd] fundamental form of M^{n+r-1} and $'H$ is a tensor field of type (1,1) associated with H. Let K and $\widetilde{K}$ stand for the curvature tensors of the hyper surface and the enveloping manifold. Then we have Gauss and Codazzi equations.

(1.15) $$\widetilde{K}(BX, BY, BZ, BU) = {}'K(X,Y,Z,U) - H(Y,Z)\, H(X,U) + H(X,Z)\, H(Y,U)$$

and

(1.16) $${}'\widetilde{K}(BX, BY, BZ, N) = (D_X H)(Y,Z) - (D_Y H)(X,Z),$$

Where $'K$ and $'\widetilde{K}$ are the associate covariant curvature tensors of M^{n+r-1} and M^{n+r}. Now let us differentiate equation (1.8) along the hyper surface and use $E_{\widetilde{X}} F = 0$ hence

$$E_{BX} B f Y = F(E_{BX} BY) - \{(D_X A)Y + A(D_X Y)\}N - A(Y) E_{BX} N.$$

In view of (1.9), (1.13) and (1.14) we get

(1.17) $$(D_X f)Y = H(X,Y)\, P + \alpha(Y)\, 'HX,$$

(1.18) $$(D_X \alpha)Y = H(X,Y)\eta - H(X, fY).$$

Covariant differentiation of (1.9) along M^{n+r-1} yields

(1.19) $$D_X P = \eta\ 'HX - 'HfX.$$

Definition 1.1 *An almost r-contact hyperbolic structure is said to be normal if*

(1.20) $$S(X,Y) = N(X,Y) + \sum_{l=1}^{r} \{(D_X \alpha)Y - (D_Y \alpha)X\}\, t^l = 0,$$

Where $$N(X,Y) = (D_{fX} f)Y - (D_{fY} f)X + f(D_Y f)X - f(D_X f)Y + \sum_{l=1}^{r} a_l [X,Y]\, t^l,$$

So that the normality condition (1.20) takes the form

$$S(X,Y) = (D_{fX} f)Y - (D_{fY} f)X + f(D_Y f)X$$
$$- f(D_X f)Y + \sum_{l=1}^{r} a_l [X,Y] t^l .$$
$$+ \sum_{l=1}^{r} \{(D_X \alpha)Y - (D_Y \alpha)X\} t^l = 0.$$

If almost r-contact hyperbolic structure induces on M^{n+r} be normal, from the last equation and from (1.17) and (1.18) we obtain

$$\alpha(X)\{'Hf - f\,'H\}Y - \alpha(Y)\{'Hf - f\,'H\}X = 0$$

(1.21) $$'Hf = f\,'H$$

Therefore it follows that [1]

(1.22) $$H(P,P) = {}'HP$$

Showing that $H(P,P)$ is an eigen value of $'H$ and the corresponding eigen vector is P. Let us denote $H(P,P)$ by τ.

Definition 1.2. *An almost r-contact hyperbolic structure is called r-hyperbolic Sasakian if*

(1.23) $$\sum_{l=1}^{r} \{(D_X \alpha_l)Y - (D_Y \alpha_l)X\} = r\,'f(X,Y).$$

We have, $'f(X,Y) = g(fX,Y)$.

More generally in a normal r-contact hyperbolic structure hyper surface of M^{n+r} we assume that [3]

(1.24) $$\sum_{l=1}^{r} \{(D_X \alpha_l)Y - (D_Y \alpha_l)X\} = r\beta\,'f(X,Y).$$

Applying (1.18) to the above equation we have

(1.25) $$'H\eta = {}'Hf = -r'\beta f.$$

Thus we obtain

(1.26) $$'HX = -r'\beta x + (\tau + r'\beta)\,\alpha(X)P,$$

Equation (1.17), (1.18), (1.19) then transform as

(1.27) $$(D_X f)Y = -r'\beta\,\{g(X,Y)P + \alpha(Y)X\} + 2(\tau + r'\beta)\,\alpha(X)\,\alpha(Y),$$

(1.28) $$(D_X \alpha)Y = r'\beta f(X,Y),$$

(1.29) $$D_X P = -(\eta - f)\, r'\beta X.$$

Let β be a constant so that from (1.27) and (1.29) we obtain

$$K(X, Y, P) = -r'^2 \beta^2 \; \eta(\alpha(Y))X - \alpha(X)Y),$$

which shows that for a normal r-contact hyperbolic structure hypersurface satisfying (1.24) and involving constant $r'\beta$, the sectional curvature with respect to a plane section containing P is $r'^2 \beta^2$.

Let us call such a structure a normal r-contact hyperbolic structure with f sectional curvature $r'^2 \beta^2$.

2. Infinitesimal Variation of a Hypersurface of an Almost *r*-contact Hyperbolic Structure Manifold

Let us take the restriction of an almost decomposable killing vector field U on the enveloping manifold of the hypersurface. According the variation of the differential of imbedding is given by [4].

(2.1) $$(\delta B)X = \in \mathrm{E}_{BX}\mathrm{U}$$

where $\in$ is infinitesimally small number. Splitting U into its tangential and normal parts as

(2.2) $$\mathrm{U} = BV + \lambda N$$

and from (1.13), (1.14) we express (2.1) as

(2.3) $$(\delta B)(X) = \in \{B(D_X V - \lambda\, 'HX) + (X\lambda + H(X,V))N\}.$$

Infinitesimal *V*ariation of N is given by [5]

(2.4) $$\delta N = \in L_U N = \in BW$$

The Lie derivative of N (i.e., $L_U N$) being orthogonal to N. Infinitesimal variation of equation (1.6) yields

$$G(BD_X V + H(X,V)N + (X\lambda)N - \lambda B\, 'HX, N) = -G(BX, BW)$$

which implies that $W = -('HV + \Lambda)$

Where Λ stands for the vector field associate to the gradient of λ. Thus we have

$$\delta N = -\in B('HV + \Lambda)$$

Now varying equation (1.8) infinitesimally, we get

$$(\delta B)(fX) + B(\delta f)X = F((\delta B)X) - (\delta N)\,\alpha(X) - \delta\alpha(X)N.$$

Making use of (1.8), (2.3) and (2.4) in it we find

$$B(\delta f)X + (\delta\alpha)(X)N = \in[\{Bf(D_X V - \lambda\,'HX)N + \alpha(D_X V - \lambda\,'HX)N + (X\lambda + H(XV)(BP + \eta N) + \alpha(X)B('HV + \Lambda)\} - \{B(D_{fX}V - \lambda\,'HfX) + (fX\lambda + H(fXV)N\}].$$

Comparing the tangential and normal components, we have

(2.5) $$(\delta f)X = \in\{f(D_X V - \lambda\,'HX) + (H(X,V) + X\lambda)P + \alpha(X)('HV + \Lambda) - D_{fX}V + \lambda\,'HfX\}.$$

and

(2.6) $$(\delta\alpha)X = \in\{f(D_X V - \lambda\,'HX) + \eta(H\lambda + H(X,V) - X\lambda - H(fX,V)\}$$

Since the derivative of f along V is given by

$$(L_V f)X = L_V(fX) - f(L_V X) = D_V(fX) - D_{fX}V - f(D_V X - D_X V).$$

Therefore equation (2.5) assumes the following form

(2.7) $$(\delta f)X = \in\{(L_V f)X + \lambda('Hf - f'H)X + X\lambda P + \alpha(X)\Lambda + 2H(X,V)P\}$$

Applying equation (1.18) and the definition

$$(L_V\alpha)X \overset{\text{def}}{=} (D_V\alpha)X + (D_X V)$$

(2.8) $$(\delta\alpha)X = \in[\{(L_V\alpha)X - \alpha\lambda\,'HX - (fX)\lambda\} + 2H(X,V)\eta + 2h(V,fX)]$$

Next varying equation (1.9) infinitesimally, we get

$$-\in FB('HV + \Lambda) = [B(\delta P) + \in\{B(D_P V - \lambda\,'HP) + P\lambda + H(P,V)N\} - \in\eta B('HV + \Lambda)].$$

Which by virtue of (1.8) and (2.3) yields

$$B\,\delta P + \in\{B(D_P V - \lambda\,'HP) + (P\lambda + H(P,V)N\} - \in\eta B('HV + \Lambda) = -\in[Bf('HV + \Lambda) + \alpha('HV + \Lambda)N],$$

whose tangential part reduces in virtue of (1.19) to the form

(2.9) $$\delta P = \in [\lambda' HP + L_U P + \Lambda(\eta - f)].$$

Again varying equation (1.5) infinitesimally, we get

(2.10) $$(\delta g)(X,Y) = G((\delta B)X, BY) + G(BX, (\delta B)Y),$$

which in virtue of (2.3) reduces to

(2.11) $$(\delta g)(X,Y) + \in \{(L_V g)(X,Y) - 2\lambda H(X,Y)\}.$$

Thus we establish the following theorem.

Theorem 2.1. *When a hyper surface of an almost r-contact hyperbolic structure manifold varied infinitesimally by means of a vector field* $U = BV + \lambda N$ *the structure tensors of almost r-contact hyperbolic structure hypersurface vary according to equations* (2.7), (2.8), (2.9) *and* (2.10).

Corollary 2.1. *When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimally tangential variation by means of BV, the variation of the induced almost r-constant hyperbolic structure tensors on the hypersurface are given by their Lie-derivatives along V.*

Corollary 2.2. *When a hypersurface of an almost r-contact hyperbolic structure manifold is given infinitesimal normal variation by means of λN, the variation of the induced almost r-contact hyperbolic structure tensors on the hyper surface are given by*

(a) $(\delta f)X = \in [\lambda('Hf - f'H)X + X\lambda P + \alpha(X)\Lambda + 2H(X,V)P],$

(b) $(\delta\alpha)X = \in [-\alpha\lambda\, 'HX - fX\lambda + 2H(X,V)\eta + 2H(V, fX)],$

(2.11) (c) $(\delta P) = \in [\lambda' HP + \Lambda(\eta - f)].$

(d) $(\delta g)(X,Y) = -2 \in \lambda H(X,Y).$

The infinitesimal variation is said to be parallel when BX and $B\overline{X}$ are both parallel equivalently and when $(\delta B)X$ is tangential to the original hyper surface. Since

$$(\delta B)X = \in [B(D_X V - \lambda\, 'HX) + (X\lambda + H(X,V)N].$$

Therefore for an infinitesimal parallel variation it is necessary and sufficient that

(2.12) $$X\lambda + H(X,V) = 0.$$

Corollary 2.3. *When a hyper surface of an almost r-contact hyperbolic structure manifold is given infinitesimal parallel variation the hypersurface variation the hypersurface*

$$\text{(2.13)}\quad \begin{aligned} &\text{(a)}\quad (\delta f)X = \in[\lambda(f\,'H - f\,'H)X + \alpha(X)\Lambda],\\ &\text{(b)}\quad (\delta\alpha)X = \in[-\alpha\lambda\,'HX],\\ &\text{(c)}\quad (\delta P) = \in\lambda\,'HP,\\ &\text{(d)}\quad (\delta g)(X,Y) = -2\in\lambda H(X,Y). \end{aligned}$$

Corollary 2.4. *Let the structure induced on a hypersurface of an almost r-contact hyperbolic structure manifold be a normal r-contact hyperbolic structure with f-sectional curvature-$\gamma'^2\beta^2$ then the infinitesimal normal parallel variation of the hypersurface makes the structure tensor vary as*

$$\text{(2.14)}\quad \begin{aligned} &(\delta f)X = \alpha(X)\Lambda,\\ &(\delta\alpha)X = -\lambda\tau P,\\ &\delta P = \in\lambda\tau P,\\ &(\delta g)(X,Y) = -2\in\lambda\{-r'\beta\, g(X,Y) + (\tau + r'\beta)\,\alpha(X)\,\alpha(Y)\}. \end{aligned}$$

3. Variation of r–Hyperbolic Sasakian Hypersurface with f – Sectional Curvature $r'^2\beta^2$

We now assume that an almost r-contact hyperbolic structure induced on the hyper surface is a r-hyperbolic Sasakian structure with f-sectional curvature $r'\beta$, we have [1]

$$\text{(3.1)}\quad H(X,\,'HY) = r'^2\beta^2 g(X,Y) + (\tau^2 + r'^2\beta^2)\,\alpha(X)\,\alpha(Y)$$

and

$$\text{(3.2)}\quad H(X,Y) = -r'\beta g(X,Y) - r'\beta(\delta g)(X,Y) + \delta(\tau + r'\beta)\,\alpha(X)\,\alpha(Y).$$

The variation in the connexions and the second fundamental form are given by [1].

$$\text{(3.3)}\quad (\delta D)(X,Y) = \in\{(L_U D)(X,Y) - (D_Y\lambda\,'H)X - (D_X\,\lambda\,'H)Y + H(X,Y) + \lambda H^*(X,Y)\}$$

where

$$g\,H^*(X,Y)Z = (D_Z H)(X,Y)$$

and

$$\text{(3.4)}\quad (\delta H)(X,Y) = \in\{(L_V H)(X,Y) - \lambda H(X,\,'HY) + XY\lambda - (D_X Y)\lambda + \lambda\,'K(N, BX, BY, N)\}$$

If the infinitesimal variation of the hyper surface are normal the variation of D would be given by [1].

$$(3.5)\quad (\delta D)(X,Y) = \in [XY\lambda - (D_X Y)\lambda + {}'K(N, BX, BY, N) - \lambda H(X\ 'HY)].$$

Varying equation (3.2) infinitesimally, we have

$$(3.6)\quad \begin{aligned}(\delta H)(X,Y) = &-(\delta\, r'\beta)\, g(X,Y) - r'\beta(\delta g)(X,Y)\\ &+ \delta(\tau + r'\beta)\,\alpha(X)\,\alpha(Y)\\ &+ (\tau + r'\beta)\{(\delta\alpha)(X)\,\alpha(Y) + \alpha(X)(\delta\alpha)(Y)\}.\end{aligned}$$

which with the help of equations (2.8), (2.9), (2.10), (3.5) and

$$(3.7)\quad \begin{aligned}(L_V H)(X,Y) = &-r'\beta(L_V g)(X,Y) + \{(L_V H)(P,P)\\ &+ 2H(L_V P,P)\}\,\alpha(X)\,\alpha(Y)\\ &+ (\tau + r'\beta)\{(L_V\alpha)(X)\alpha(Y)\\ &+ \alpha(X)(L_V\alpha)(Y)\}.\end{aligned}$$

becomes

$$(3.8)\quad \begin{aligned}&\in\{X, Y\lambda - (D_X Y)\lambda + \lambda\ {}'\tilde{K}(N, BX, BY, N) - \lambda H(X\ 'HY)\}\\ &= -2\, r\beta \in \lambda H(X,Y) + \in (PP\lambda - (D_P P)\lambda\\ &- \lambda H(P,HP) - 2H(P, \lambda HP - \Lambda(\eta - f)\} + \delta r'\beta / \in\}(X)\,\alpha(Y)\\ &+ \in (\tau + r'\beta)(-\alpha\lambda\ 'HX - f X\lambda + 2H(X,V)\,\eta\\ &+ 2H(V, fX\}\,\alpha(Y) + (-\alpha\lambda\ 'HY + fY\lambda\\ &+ 2H(Y,V\} + 2H(V, fY\}\,\alpha(X),\end{aligned}$$

Conversely if λ satisfies the differential equation (3.8) then by retreating the steps we get (3.3).

Hence we have the following theorem

Theorem 3.1. *In order that for an infinitesimal variation (2.1) may have the a r-hyperbolic Sasakian hypersurface with f-sectional curvature* $-r'^2\beta^2$ *in a r-hyperbolic Sasakian with f-sectional curvature* $-r'^2\beta^2 - \delta r'^2\beta^2$, *it is necessary and sufficient that the function* λ *satisfies the relation*

$$\begin{aligned}&\in\{XY\lambda - (D_X Y)\lambda + \lambda\{\ 'K(N, BX, BY, N)\} + \gamma'^2\beta^2\ (g(X,Y) - \alpha(X) - \alpha(Y))\\ &+ (PP\lambda - D_P P)\lambda)\,\alpha(X)\,\alpha(Y) + (\tau + r'\beta)\{-fX\lambda\,\alpha(Y)\\ &- fY\lambda\,\alpha(X)\}\, f\{2H(X,V)\,\tau + 2HV, fX)\}\,\alpha(Y)\\ &+ \{2H(Y,V) + 2H(V, fY)\}\,\alpha(X)]\\ &= \delta r'\beta(\alpha(X)\,\alpha(Y) - g(X,Y)).\end{aligned}$$

Corollary 3.1. *The infinitesimal normal parallel variation carries a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature* $-r'^2\beta^2$ *to a normal r-hyperbolic Sasakian hypersurface with f-sectional curvature* $-r'^2\beta^2 - \delta r'^2\beta^2$ *if and only if*

$$(3.9) \qquad \lambda \in \{ 'K(N, BX, BY, N)\} + r'^2\beta^2 \,(g(X,Y) - \alpha(X)\,X(Y)\}$$
$$= \{\alpha(X)\,\alpha(Y) - g(X,Y)\,\delta r'\beta$$

Corollary 3.2. *If the enveloping manifold of corollary* (3.1) *be flat the condition reduced to* $\delta r'\beta = -\lambda \in r'^2\beta^2$.

Hence the proof is obvious.

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