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Invariant and non invariant hypersurfaces of almost Lorentzian para contact manifolds

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Goldberg, S. I. and Yano studied and defined Noninvariant Hypersurfaces of almost contact manifolds and has become subject of sufficient interest and Sato (1976) studied about a structure similar to almost contact structure. In present paper our aim is to study Invariant and Noninvariant Hypersurfaces of almost Lorentzian Para contact manifolds.

Introduction. Let V_n be an n –dimensional differentiable manifold endowed with a tensor field ϕ of type (1,1) a vector field U and a 1–form u such that

(1.1) $$\phi^2 = 1 + u \otimes U,\ u(U) = -1,\ \phi\ U = 0$$
$$u \circ \phi = 0, \quad \text{rank}\ \phi = n-1.$$

Then V_n is said to have an almost Lorentzian Para contact structure. If in V_n there exist a Remannian metric g such that

(1.2) $$u(X) = g(X,U),$$
$$g(\phi X, \phi Y) = g(X,Y) + u(X)\, u(Y),$$

Then V_n is said to have an almost Lorentzian Para contact metric structure [4]. We say that the almost Lorentzian Para contact structure is normal if

(1.3) $$[\phi,\phi] - U \otimes du = 0.$$

Where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ.

An almost Lorentzian Para contact metric structure is said to be Lorentzian Para-Sasakian,

if

(1.4) $$(D_X \phi)(Y) = u(Y)X + 2\, u(X)u(Y)\, U + g(X,Y)U,$$

where D denotes the Riemannian connexion of g([1],[2]). An almost Lorentzian Para contact metric manifold is said to be a closed almost Lorentzian Para contact metric manifold if u is closed. Further if,

(1.5) $$D_X U = \phi X.$$

Then it is called a Lorentzian Para contact metric manifold ([2], [4]). It is observed that a Lorentzian Para-Sasakian manifold is a Lorentzian Para contact metric manifold.

An almost product manifold is a differentiable manifold which has a (1,1) tensor field J satisfying the condition,

(1.6) $$J^2 = I.$$

Moreover if there exist a Riemannian metric g such that,

(1.7) $$g(JX, JY) = g(X, Y),$$

then it is called an almost product metric manifold. Let D be the Riemannian connexion of g. then the manifold is said to be an almost product almost decomposable manifold if,

(1.8) $$(D_X, J)(Y) = 0$$

Consider an almost Lorentzian Para contact manifold V_n and let V_m be an orientable hypersurface of V_n, and B the differential of the immersion I of V_m into V_n. Let X,Y and Z be tangent to V_m and C a unit normal vector.

Then we have

(1.9) $$\phi BX = BFX + \alpha(X)\, C.$$

Where F is a (1,1) tensor field, and α a 1 - form on V_m. If $\alpha \neq 0$, then V_m is called a non-invariant hypersurface of V_n. If α is identically zero, then V_m is said to be an invariant hypersurface, that is, the tangent space of V_m is invariant under ϕ [3].

The metric g of an almost Lorentzian Para contact metric manifold induces a Riemannian metric G on the Hypersurface V_m given by,

(1.10) $$G(X,Y) = g(BX, BY).$$

Further the symmetric affine connexion D on V_n induces a symmetric affine connexion $\overline{D}$ on the hypersurface V_m such that,

(1.11) $$D_{BX}\, C = B(\overline{D}_X\, Y) + h(X,Y)C,$$

where h is a symmetric tensor of type (0,2) called the second fundamental form of the hypersurface V_m. We have,

(1.12) $$D_{BX}\, C = -BHX + W(X)C,$$

where W is a 1 - form on V_m defining the connexion an affine normal bundle and H is a (1,1) tensor field on V_m such that $g(HX, Y) = h(X, Y)$.

2. Noninvariant Hypersurfaces of Almost Lorentzian Para Contact Manifolds.

Let V_n be an almost Lorentzian Para Contact Manifold with the structure tensors (ϕ, U, u), and V_m a non-invariant hypersurface of V_n. In what follows we assume that U is nowhere tangent to V_m and so we can take $C = U$, then (1.9) takes the form,

(2.1) $$\phi BX = BFX + \alpha(X)U.$$

Theorem 1. *If V_m is a noninvariant hypersurface of an almost Lorentzian Para Contact manifold V_n with U nowhere tangent to V_m, then V_m admits an almost product structure.*

Proof : Applying ϕ to both sides of (2.1), we get

$$\phi^2 BX = \phi\ BFX + \phi(\alpha(X)\ Up).$$

From (1.1) & (2.1) we have

$$BX + u(BX)U = BF^2X + \alpha(FX)U.$$

Now equating the co-efficient of above equation we have,

$$F^2X = X, \tag{2.2}$$

$$u(BX) = \alpha\ (FX). \tag{2.3}$$

Thus F acts as an almost product structure on V_m.

Theorem 2. *If V_m is a noninvariant hypersurface of an almost Lorentzian Para contact metric manifold V_n (ϕ, U, u, g), then V_m is an almost product metric manifold.*

Proof : From Theorem (1) it follows that V_m has an almost product structure F. Let G be the induced metric in V_m, that is,

$$g(BX,BY) = G(X,Y)$$

Now we define a metric on V_m by

$$G^*(X,Y) = G(X,Y) - \alpha(X)\alpha(Y).$$

Then we have,

$$G^*(FX,FY) = G(FX,FY) - \alpha(FX)\ \alpha(FY).$$

Applying the condition of equation (1.10), (2.1) & (2.3) in above equation we have,

$$G^*(FX,FY) = g(BFX,\ BFY) - u(BX)\ u(BY),$$

$$= g(\phi\, BX - \alpha(X)U, \phi\, BY - \alpha(Y)U) - u(BX)\ u(BY),$$

$$= g(\phi\, BX, \phi\, BY) - \alpha(X)\alpha(Y) - u(BX)\ u(BY)$$

Applying the condition of equation (1.2) in above equation

$$= g\ (BX,BY) + u\ (BX)\ u\ (BY) - \alpha\ (X)\ \alpha\ (Y) - u\ (BX)\ u\ (BY)\ ,$$

$$= g\ (BX,BY) - \alpha\ (X)\alpha\ (Y),$$

$$= G^*\ (X,Y).$$

Hence G^* is the metric which makes V_m an almost product metric manifold.

Theorem 3. *Let V_m be a noninvariant hypersurface of Lorentzian Para contact metric manifold V_n then,*

$$\text{(a) } F = -H, \quad \text{(b) } \alpha = W \tag{2.4}$$

Proof : Since V_n is a Lorentzian Para contact metric manifold, we have,

$$D_{BX}\ U_P = \phi\ BX.$$

Using (1.12) and (2.1) we have

$$-BHX + w(X)U = BFX + \alpha\ (X)U.$$

Which gives

$$F = -H,$$
$$\& \quad \alpha = W.$$

Theorem 4. *If V_m is a noninvariant hypersurface of a Lorentzian Para-Sasakian manifold V_n then,*

$$(\overline{D}_X F)(Y) = \alpha(FY)X - \alpha(Y)FX,$$
$$g(BX, BY) = h(X, FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) - 2\alpha(FX)\alpha(FY)$$

Proof : We know that

$$(D_{BX}\phi)(BY) = D_{BX}\phi BY - \phi(D_{BX}BY).$$

Using equation (1.4), (1.9) & (1.11) in the above equation we get,

$$U(BY)BX + 2u(BX)u(BY)U + g(BX, BY)U = D_{BX}(BFY + \alpha(Y)U) - \phi(B\overline{D}_X Y + h(X,Y)U)$$

$$= D_{BX}BFY + D_{BX}\alpha(Y)U - \phi BD_X Y,$$
$$= B(\overline{D}_X FY) + h(X, FY)U + (D_{BX}\alpha(Y)) + \alpha(Y)D_{BX}U - BF\overline{D}_X Y - (\overline{D}_X Y)U,$$
$$= B(\overline{D}_X F)(Y) + \{h(X, FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y)\}U + \alpha(Y)D_{BX}U.$$

In consequence of equation (1.5) we have,

$$= B(\overline{D}_X F)(Y) + \{h(X, FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y)\}U + \alpha(Y)\phi BX,$$
$$= B(\overline{D}_X F)(Y) + \{h(X, FY) + D_{BX}\alpha(Y) - \alpha(\overline{D}_X Y) + \alpha(Y)\alpha(X)\}U + \alpha(Y)BFX,$$

equating the components we get

$$(\overline{D}_X F)(Y) = -\alpha(Y)FX + u(BY)X,$$

and

$$h(X, FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) = 2u(BX)u(BY) + g(BX, BY).$$

From equation (2.3) we have

$$(\overline{D}_X F)(Y) = \alpha(FY)X - \alpha(Y)FX,$$
$$h(X, FY) + (\overline{D}_X \alpha)(Y) + \alpha(X)\alpha(Y) = 2\alpha(FX)\alpha(FY) + g(BX, BY).$$

As an immediate consequence we have the following:

COROLLARY : Let V_m be a noninvariant hypersurface of Lorentzian Para-Sasakian manifold V_n with the induced almost product structure F. Then V_m is an almost product almost decomposable manifold if and only if

$$\alpha(Y)FX = \alpha(FY)X.$$

3. Invariant hypersurface of almost Lorentzian paracontact manifolds.

Let $V_n(\phi, U, u)$ be an almost Lorentzian Para contact manifold and let V_m be an invariant hypersurface of V_n. Then, equation (1.9) becomes

$$\phi BX = BFX.$$

In what follows we study the invariant hypersurface with the following conditions:

(a) When U is nowhere tangent to V_m.
(b) When U is everywhere tangent to V_m.

When U is nowhere tangent to V_m.

Theorem 5. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. Then V_m is an almost product manifold with $u(BX) = 0$.*

Proof : The proof follows from theorem (1), for invariant hypersurface $\alpha = 0$, then we will get

(3.1) $$u(BX) = 0.$$

Theorem 6. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. If V_n is normal then the almost product structure induced on V_m is integrable.*

Proof : We know that,

$$[\phi,\phi](BX,BY)=\phi^2[BX,BY]+[\phi BX,\phi BY]-\phi[\phi BX,BY]-\phi[BX,\phi BY].$$

Using equation (3.1) and $B[X,Y] = [BX, BY]$ in above equation we have

$$[\phi,\phi](BX,BY)=BF^2[X,Y]+B[FX,FY]-BF[FX,Y]-BF[X,FY],$$
$$= B[F,F](X,Y),$$

further we have

$$du(BX,BY) = BX.\ u(BY) - BYu(BX) - u(B[X,Y])$$
$$= 0.$$

Thus we can write,

$$[\phi,\phi](BX,BY) - du(BX,BY)U = B[F,F](X,Y).$$

Hence the theorem is proved.

Theorem 7. *An invariant hypersurface V_m of a Lorentzian Para-Sasakian manifold V_n is an almost product almost decomposable manifold.*

Proof : From theorem (5) it follows that V_m is an almost product manifold. Further, theorem -3 gives that it is metric also. Now from (1.11) we have,

$$D_{BX}BFY = B\overline{D}_X FY + h(X,FY)U,$$
$$= B[(\overline{D}_X F)(Y) + F\overline{D}_X Y] + h(X,FY)U,$$
$$D_{BX}BFY = B(\overline{D}_X F)(Y) + BF\overline{D}_X Y + h(X,FY)U,$$

(3.2) $$D_{BX}BFY - BF\ \overline{D}_X Y = B(\overline{D}_X F)(Y) + h(X,FY)U,$$

From equation (1.4) we have

$$(D_X\phi)(Y) = u(Y)X + 2u(X)u(Y) + g(X,Y)U,$$

$$(D_{BX}\phi)(BY) = u(BY)BX + 2u(BX)\,u(BY) + g(BX, BY)U,$$
$$(D_{BX}\phi)(BY) = g(BX, BY)\,U,$$
$$D_{BX}\,\phi\, BY - \phi D_{BX}\,\phi\, BY = g(BX, BY)U,$$
$$D_{BX}\phi\, BFY - \phi[B\overline{D}_X Y + h(X,Y)U] = g(BX, BY)U,$$
$$D_{BX}\,\phi BFY - \phi\, B\overline{D}_X Y = g(BX, BY)\,U,$$
$$D_{BX}\,\phi\, BFY - BF\,\overline{D}_X Y = g(BX, BY)U.$$

From equation (3.2) we have,

$$g(BX, BY)\,U = B((\overline{D}_X F)(Y)) + h(X, FY)\,U.$$

Hence

$$(\overline{D}_X F)(Y) = 0.$$
$$g\,(BX, BY) = h(X, FY).$$

Which completes the proof.

When U is everywhere tangent to V_m.

Theorem 8. *Let V_m be an invariant hypersurface of an almost Lorentzian Para contact manifold V_n. Then V_m is almost Lorentzian Para contact manifold. Further, if V_n is normal then V_m is normal.*

Proof : Since U is everywhere tangent to V_m then in a unique vector field U^* such that

$$BU^* = U. \text{ Set}$$
$$u^*(BX) = u(BX)$$

Then u^* is a 1-form on V_m. Further, we have,

$$BFX = \phi\; BX,$$

Which implies

$$BF^2 X = \phi^2 BX = BX + u^*(X)\, BU^*,$$

That is

$$F^2 X = X + u^*(X)U^*.$$

Also

$$u^*(FX) = u(B\,FX) = u(\phi\, BX) = 0,$$
$$u^*(U^*) = u(BU^*) = u(U) = -1,$$

and

$$BF(U^*) = \phi\, BU^* = \phi\, U = 0.$$

Which gives that,

$$F(U^*) = 0.$$

Thus V_m be is an almost Lorentzian Para contact manifold with the structure tensors (F, U^*, u^*).

Finally we have,

$$N(BX, BY) = [\phi, \phi]\,(BX, BY) - du\,(BX, BY)U,$$

$$= \phi^2 B[X,Y] + [\phi BX, \phi BY] - \phi[BX, \phi BY] - \phi[\phi BX, BY]$$
$$-\{BX.u(BY) + BY.u(BX) + u(B[X,Y])U\},$$
$$= BF^2[X,Y] + [BFX, BFY] - BF[X,FY] - BF[FX,Y]$$
$$-\{BX.u^*(Y) + B.Y.u^*(X) + U^*([X,Y]\} BU^*,$$
$$= B\{[F,F](X,Y) - du^*(X,Y)U^*\}.$$

Hence, if V_n normal, then V_m. is also normal.

Theorem 9. *If V_m is an invariant hypersurface of a Lorentzian Para contact metric manifold V_n. Then V_m is also Lorentzian Para contact metric manifold.*

Proof : From theorem 8 it follows that V_m is an almost Lorentzian Para contact manifold with structure tensors (F, U^*, u^*). Let g^* be the induced metric on V_m. Then we have,

$$g^*(FX, FY) = g(BFX, BFY) = g(\phi BX, \phi BY)$$

Further we easily show that u^* is closed. Finally, since V_n is a Lorentzian Para contact metric manifold, we have-

$$D_{BX}U = \phi BX,$$

that is

$$D_{BX}BU^* = BFX,$$

Which is in consequence of (1.11) becomes,

$$B\overline{D}_X U^* + h(X, U^*)C = BFX,$$

Where C is normal to V_n.
Equating the components of above equation we have,

$$\overline{D}_X U^* = FX,$$
$$\& \ h(X,U^*) = 0$$

Which completes the proof.

Theorem 10. *Let V_m is an invariant hypersurface of a Lorentzian Para Sasakian manifold V_n. Then V_m is a Lorentzian Para Sasakian manifold.*

Proof : we have proved that V_m is a Lorentzian Para contact manifold, Now we have,

$$B(\overline{D}_X F)(Y) = B\overline{D}_X FY - BF(\overline{D}_X Y).$$

Which is in consequence of equation (1.11) and (3.1) becomes,

$$B(\overline{D}_X F)(Y) = D_{BX} BFY - h(X,FY)C - \phi(B\overline{D}_X Y),$$
$$= D_{BX}\phi BY - \phi(D_{BX} BY - h(X,Y)C) - h(X,FY)C,$$
$$= (D_{BX}\phi) BY + \phi(h(X,Y)C) - h(X,FY).C.$$

Using equation (1.4) in above equation we have,

$$B(\overline{D}_X F)(Y) = u(BY)BX + 2u(BX)u(BY)U$$
$$+ g(BX, BY)U + \phi(h(X,Y)C) - h(X,FY)C,$$

$$= B[u^*(Y)X + 2u^*(X)\, u^*(Y)U^* + g^*(X,Y)U^*] + h(X,Y)\phi C - h(X, FY)\, C.$$

Since C and U are linearly independent unit vectors, C can be thought of as eigen vector of ϕ corresponding to eigen + or – and $\phi C = \pm$ C.
Equating the tangential and normal components we have

$$(\overline{D}_X F)(Y) = u^*(Y)\, X + 2u^*(X)\, \dot{u}^*(Y)U^* + g^*(X,Y)U^*.$$
$$h(X,Y)\phi C = h(X,FY)\, C.$$

Hence V_m is Lorentzian Para Sasakian manifold, Now we have.

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