The Nepali Math. Sc. Report
Vol. 27. No.1 & 2. 2007.

Köthe–Toeplitz duals

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Abstract: We define sequence space $(\overline{\ell(p)})_t$. We also establish Köthe-Toeplitz Duals of $(\overline{\ell(p)})_t$.

1. Introduction.

The following definitions and notations will be useful in our discussion: ℓ_p = the space of sequences $x = (x_k)$ with absolutely p-summable series $(1 \le p < \infty)$. If $p = (p_k)$ is a bounded sequence of strictly positive real numbers, then

$$\ell(p) = \left\{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \right\}$$

Let $t = (t_k)$ be any fixed sequence of non-zero complex numbers satisfying $\liminf (t_k)^{1/k} = \gamma \, (0 < \gamma \le \infty)$ and let

$$\overline{\ell(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty \right\}$$

where $t_k(x) = \sum_{i=1}^{k} x_i$, then we define

$$\left(\overline{\ell(p)}\right)_t = \left\{ x = (x_k) : (t_k x_k) \in \overline{\ell(p)} \right\}$$

2. Topological properties of $(\overline{\ell(p)})_t$.

Theorem 2.1. *$(\overline{\ell(p)})_t$ is a complete paranormed space.*

Proof : As $(0) \in (\overline{\ell(p)})_t$, $(\overline{\ell(p)})_t \neq 0$, it is easy to verify that it is a linear space. And also it is clear that the function defined by $g^*(x) = g(tx)$ where g is the paranorm in $\overline{\ell(p)}$ satisfying that $g^*(0) = 0$, $g^*(x) = g^*(-x)$ and $g^*(x+y) \leq g^*(x) + g^*(y)$. Clearly, $\lambda_n \to \lambda$ in C and $g^*(x^n - x) \to 0$ as $n \to \infty$ imply that $g^*(\lambda_n x^n - \lambda x) \to 0$ as $(n \to \infty)$ where $x^n = (x_k^n) = \left(x_1^n, x_2^n, x_3^n, \ldots, x_k^n, \ldots\right)$ and $x = (x_k)$, hence g^* is a paranorm in $(\overline{\ell(p)})_t$. To show that $(\overline{\ell(p)})_t$ is complete, let (x^n) be a Cauchy sequence in $(\overline{\ell(p)})_t$ where

$$x^n = \left(x_1^n, x_2^n, x_3^n, \ldots, x_k^n, \ldots \right) \in (\overline{\ell(p)})_t$$

Then $(tx^n) = \left((t_k x_k^1), (t_k x_k^2) \ldots \right)$ is a Cauchy sequence in $\overline{\ell(p)}$. As $\overline{\ell(p)}$ is complete, so it converges to (z_k) (say). Let $z_k = t_k x_k$ so that $x_k = t_k^{-1} z_k$. Then (tx^n) converges to $(t_k x_k) \in \overline{\ell(p)}$. Hence, $g\left(t_k x_k^n - t_k x_k\right) = g\left(t(x^n - x)\right) \to 0 \ (n \to \infty)$ implies that $g^*(x^n - x) \to 0$ as $n \to \infty$. Therefore x^n is convergent, and $(\overline{\ell(p)})_t$ is a complete paranormed space.

Corollary 2.1. *$\overline{\ell}_p$ is a Banach space for $(1 \leq p < \infty)$, normed by*

$$\| x \| = \left(\sum_{k=1}^{\infty} | t_k(x) |^p \right)^{\frac{1}{p}}$$

Here the norm in $\left(\overline{\ell(p)} \right)_t$ is defined by $s \| x \|_t = \| (t_k x_k) \|$. So it is also a Banach space.

Corollary 2.2. *$\overline{\ell}_2$ is a Hilbert space with inner product* $\langle x, y \rangle = \sum_{k=1}^{\infty} t_k(x) \overline{t_k(y)}$

Here the inner product in $(\overline{\ell}_2)_t$ is defined by $\langle x, y \rangle_t = \langle (t_k x_k), (t_k y_k) \rangle$ Hence it is a Hilbert space.

Theorem 2.2. *If z be a closed subset of $\overline{\ell(p)}$, then z_t is a closed subset of. $(\overline{\ell(p)})_t$,*

Proof. Since $z \subset (\overline{\ell(p)})$, $z_t \subset (\overline{\ell(p)})_t$ (obvious). Now let $x \in (\overline{z}_t)$ where $\overline{z}_t$ stands for the closure of z_t, then there exists a sequence $(x^n) \subset z_t$ such that (x^n) converges to x. This implies that $g^*(x^n - x) = g^*\left\{(t_k x_k^n) - (t_k x_k)\right\} \to 0 (n \to \infty)$ in z_t. Thus,

$$g^*\left\{(t_k x_k^n) - (t_k x_k)\right\} \to 0 \ (n \to \infty) \text{ in } z.$$

Hence $(t_k x_k)$ is the limit of a sequence of points in z and $(t_k x_k) \in (\bar{z})$ which yields $x \in (\bar{z})_t$. Conversely, if $x \in (\bar{z})_t$ then $x \in (\bar{z}_t)$, since z is closed, that is $\bar{z} = z$. Therefore $(\bar{z}_k) = (\bar{z})_t = z_t$, hence z_t is closed in $(\overline{\ell(p)})_t$.

3. Köthe-Toeplitz Duals of $(\overline{\ell(p)})_t$.

Definition: *Let X be a sequence space. We define:*

(i) $$X^{\alpha} = \left\{ a = (a_k) : \sum_k |a_k x_k| < \infty \text{ for all } x \in X \right\}$$

(ii) $$X^{\beta} = \left\{ a = (a_k) : \sum_k |a_k x_k| \text{ converges for all } x \in X \right\}$$

(iii) $$X^{\gamma} = \left\{ a = (a_k) : \sup \left| \sum_{k=1}^{n} a_k x_k \right| < \infty \text{ for all } x \in X \right\}$$

Then $X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the α^-, β^- and γ^- dual spaces of X respectively. In [3] the author has established the β dual of $\overline{\ell(p)}$. Here we establish the α^-, β^- and γ^- duals of $(\overline{\ell(p)})_t$.

Theorem 3.1. *Let $\lambda = \alpha, \beta, \gamma$. Then*

(i) $$\left(\left(\overline{\ell(p)} \right)_t \right)^{\lambda} = \left\{ a = (a_k) : \left(\frac{a_k}{t_k} \right) \in \ell(p) \right\} = \left(\left(\overline{\ell(p)} \right)^{\lambda} \right)_t$$

(ii) $$\left(\left(\overline{\ell(p)} \right)_t \right)^{\lambda\lambda} = \left\{ x = (x_k) : (t_k x_k) \in \left(\left(\overline{\ell(p)} \right)^{\lambda\lambda} \right)_t \right\}$$

Proof: (i) Let $\lambda = \beta$ and $a \in \alpha$ and $D = \left\{ a = (a_k) : \left(\frac{a_k}{t_k} \right) \in \left(\overline{\ell(p)} \right)^{\alpha} \right\}$

We show that $\left(\left(\overline{\ell(p)} \right)_t \right)^{\alpha} = D$

Let $a \in \left(\left(\overline{\ell(p)} \right)_t \right)^{\alpha}$ then $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for every $x \in \left(\overline{\ell(p)} \right)_t$, so that

$$\sum_{k=1}^{\infty} \left| \frac{a_k}{t_k} t_k x_k \right| = \sum_{k=1}^{\infty} |a_k x_k| < \infty$$

Since $(t_k x_k) \in \overline{\ell(p)}$ it follows that $\left(\frac{a_k}{t_k} \right) \in \left(\overline{\ell(p)} \right)^{\alpha}$ implies that $a \in D$.

Hence, $\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha \subset D$. Conversely, if $a \in D$ and $x \in \left(\overline{\ell(p)}\right)_t$, then, $\left(\dfrac{a_k}{t_k}\right) \in \left(\overline{\ell(p)}\right)^\alpha$

and $(t_k x_k) \in \overline{\ell(p)}$ so that. $\sum_{k=1}^{\infty} |a_k x_k| = \sum_{k=1}^{\infty} \left| \dfrac{a_k}{t_k} t_k x_k \right| < \infty$..

Since, $x \in \left(\overline{\ell(p)}\right)_t$, it follows that $a \in \left(\overline{\ell(p)}\right)^\alpha$. Hence $D \subset \left(\left(\overline{\ell(p)}\right)_t\right)^\alpha$

Thus, $\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha = \left(\left(\overline{\ell(p)}\right)^\alpha\right)_t$.

Similar results hold for $\lambda = \beta$ or γ as well.

(ii) Let $\lambda = \alpha$ and let $\left(\overline{\ell(p)}\right)^{\alpha\alpha}$ exist. Then

$$\left(\left(\overline{\ell(p)}\right)_t\right)^{\alpha\alpha} = \left[\left(\left(\overline{\ell(p)}\right)_t\right)^\alpha\right]^\alpha$$

$$\left[\left(\left(\overline{\ell(p)}\right)^\alpha\right)_t\right]^\alpha = \left(\left(\overline{\ell(p)}\right)^{\alpha\alpha}\right)_t$$

For $\lambda = \beta$ or γ, the proof is same.

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