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Lorentz Equation For Constant Electromagnetic Fields

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Abstract: We show that the Lorentz equation admits exact integration when the external electromagnetic field is constant. Our process uses the eigenvectors of the Faraday tensor and a 2nd order differential equation for the tangent vector to world line, which originates a method more simple comparated with the techniques of Plebañski [1], Synge [2] and Piña [3].

I. Introduction:

The trajectory of a particle in Minkowski space is given by a sequence of events $x_r(s)$:

(1) $$x_1 = x,\ x_2 = y,\ x_3 = z,\ x_4 = it,\ i = \sqrt{-1}$$

where s is the proper time and the light velocity is equal to one. Then the corresponding velocity, acceleration and superacceleration are (we shall employ the quantities and notation of Synge [2,4,5]) ;

(2) $$\lambda_r = \frac{dx_r}{ds},\ \mu_r = \frac{d\lambda_r}{ds},\ \nu_r = \frac{d\mu_r}{ds}$$

$$\lambda_r\lambda_r = -1,\ \ \mu_r\lambda_r = 0,\ \ \lambda_r\nu_r = -\mu_r\mu_r$$

If the particle has a charge e, then the Lorentz equation [4,6,7] :

(3) $$\mu_r = bF_{rj}\lambda_j,\ \ b = \frac{e}{m}$$

indicate its interaction with an external electromagnetic field characterized by the Faraday tensor [4,8] :

$$(F_{ab}) = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{4}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic vectors, respectively. In Sec 2 we introduce the dual tensor of (4) which permits to construct the classification of F_{ab} as proposed by Synge [2] and Piña [3], with great relevance in the study of the solutions for (3).

Our problem is to integrate the Lorentz equation when F_{ab} is constant, which it may be realized through the following technique :

a) Algebraical Process

Synge [2,9–11] showed that the curvatures K_r , $r = 1, 2, 3$ of the world line are constants, and from Frenet-Serret formulae [10-12] he obtains a 4th order differential equation for the velocity λ_r :

$$\frac{d^4}{ds^4}\lambda_r + \left(k_2^2 + k_3^2 - k_1^2\right)\frac{d^2}{ds^2}\lambda_r - k_1^2\, k_3^2\, \lambda_r = 0 \tag{5}$$

whose solutions depend of the corresponding Euler's characteristic equation. Therefore,

$$x_r(s) = x_r(0) + \int_0^s \lambda_r(\theta)\, d\theta \tag{6}$$

gives us the path in terms of the initial conditions $x_a(0)$ and $\lambda_a(0)$.

b) Tensorial Method

It is immediate the integration of (3) :

$$\lambda_r(s) = \exp(bsF_{rc})\, \lambda_c(0). \tag{7}$$

thus Plebañski [1] and Piña [3] indicate how to calculate the exponential function of an antisymmetric matrix (or tensor). This determines $\lambda_r(s)$ and then (6) again gives the trajectory.

In Sec. 3 we exhibit a new process (named algebraical-tensorial method) to resolve (3), which it uses the proper values and eigenvectors of Faraday tensor.

2. Algebraic Classification F_{ab}

Using the totally antisymmetric symbol $\in_{armn}$ of Levi-Civita we can construct the dual tensor of Faraday [4, 8] :

$$ {}^*F_{ac} = \frac{1}{2} \in_{acmn} F_{mn} , \tag{8} $$

with matricial expression

$$ ({}^*F_{ac}) = \begin{pmatrix} 0 & E_3 & -E_2 & iB_1 \\ -E_3 & 0 & E_1 & iB_2 \\ E_2 & -E_1 & 0 & iB_3 \\ -iB_1 & -iB_2 & -iB_3 & 0 \end{pmatrix} \tag{9} $$

Any antisymmetric tensor and its dual satisfy the identities [1,3,8,13-15]:

$$ F_{ra}F_{rc} - {}^*F_{ra}\, {}^*F_{rc} = \frac{I_1}{2}\delta_{ac} , \quad {}^*F_{ra}F_{rc} = \frac{I_2}{4}\delta_{ac} \tag{10} $$

where :

$$ I_1 = F_{ac}F_{ac} = 2\left(E^2 - B^2 \right), \quad I_2 = {}^*F_{ac}F_{ac} = 4\vec{E} \cdot \vec{B} \tag{11} $$

are the unique invariants (under Lorentz transformations) of F_{ij} , which leads to the Synge classification [2, 8, 10] for the electromagnetic field:

$$ \begin{aligned} &\text{Type A} : I_2 \neq 0 \\ &\text{Type B} : I_1 < 0 ,\ I_2 = 0 \\ &\text{Type C} : I_1 = 0 ,\ I_2 = 0 \text{ Null field} \\ &\text{Type D} : I_1 > 0 ,\ I_2 = 0 . \end{aligned} \tag{12} $$

The cases indicated in (12) are of interest because the properties of the world line depend of the Type for F_{ac} plus the initial conditions.

We also can write (11) in the form of Piña [3]:

$$ I_1 = 2HCos\gamma , \quad I_2 = 2HSin\gamma , \quad H \geq 0, \quad 0 \leq \gamma < 2\pi , \tag{13} $$

thus (12) is reduced to :

$$ \begin{aligned} &\text{Type A} : \gamma \neq 0, \pi , \qquad \text{Type B} : \gamma = \pi , \\ &\text{Type C} : \text{H} = 0, \quad \text{Type D} : \gamma = 0. \end{aligned} \tag{14} $$

The electromagnetic field is non-null if I_1 or/and I_2 are differents to zero, then in this case there are [4,13, 16-18] two null eigenvectors γ_r and η_r with proper values $\pm\lambda$:

$$F_{rc}\gamma_c = \lambda\gamma_r,\ \ F_{rc}\eta_c = -\lambda\eta_r,\ \ \gamma_r\gamma_r = \eta_r\eta_r = 0,$$

(15) $$\tilde{\lambda} = \frac{1}{4}\left(I_1^2 + I_2^2\right)^{1/2} = \frac{H}{2} > 0,$$

$$\lambda = \left(\tilde{\lambda} - \frac{I_1}{4}\right)^{1/2} = \sqrt{H}\ Sin\,\frac{\gamma}{2} > 0,$$

which permits to express F_{ij} and its dual in terms of their null principal directions :

$$F_{rc} = \frac{1}{\gamma_a\eta_a}\left[\lambda(\gamma_r\eta_c - \gamma_c\eta_r) + i\beta \in_{rcmn} \gamma_m \eta_n\right],$$

(16) $$^*F_{rc} = \frac{1}{\gamma_a\eta_a}\left[-\beta(\gamma_r\eta_c - \gamma_c\eta_r) + i\lambda \in_{rcmn} \gamma_m \eta_n\right],$$

$$\beta = \in\left(\tilde{\lambda} + \frac{I_1}{4}\right)^{1/2} = \in\sqrt{H}\ \ Cos\,\frac{\gamma}{2} \geq 0,\ \ \in = \pm 1.$$

We note that always $\gamma_a\eta_a \neq 0$ because these null eigenvectors have linear independence

3. Algebraical–Tensorial method

This method is not explicitly in the literature, and we consider that it is more simple and elementary than the processes of [1-3] because it only involve relations very known in tensorial algebra.

In fact, if we employ (10) in (3) results a 2nd order differential equation for the acceleration :

(17) $$\frac{d^2}{ds^2}\mu_r + \frac{b^2}{2} I_1\,\mu_r = -\frac{b^3}{4} I_2\, {}^*F_{rc}\,\lambda_c$$

which permits to study easily the types B,C and D because they have $I_2 = 0$, and thus one integration gives us :

(18) $$\frac{d^2}{ds^2}\lambda_r + \frac{b^2}{2} I_1\,\lambda_r = \nu_r(0) + \frac{b^2}{2} I_1\,\lambda_r(0)$$

with more simplicity than (5). The nature of the characteristic roots α of (18) depends of the invariant I_1, therefore :

Type B:

$$\alpha = \pm\, b\lambda\,, \qquad \lambda = \sqrt{-\frac{I_1}{2}} > 0.$$

Then the 4-velocity solution of (18) is :

(19) $$\lambda_r(s) = \lambda_r(0) - \frac{1}{b^2\lambda^2}\, \nu_r(0) + A_r e^{b\lambda S} + B_r e^{-b\lambda S}$$

where the integration's constants A_r and B_r are determined in terms of the initial conditions :

(20) $$\mu_r(0) = bF_{rc}\lambda_c(0),\ \ \nu_r(0) = b^2\, F_{rc}\, F_{cn}\lambda_n(0)$$

If we remember that $e^{\pm b\lambda S} = Cosh\,(b\lambda s) \pm\ Sinh\,(b\lambda s)$, then (19) adopts the form

$$\lambda_r(s) = \lambda_r(0) + \frac{1}{b\lambda}\, \mu_r(0) Sinh(b\lambda s) + \frac{1}{b^2\lambda^2}\, \nu_r(0)\,(Cosh(b\lambda s) - 1),$$

and thus (6) implies the path :

(21) $$x_r = x_r(0) + \left[s\,\delta_{rn} + \frac{1}{b^2\lambda^2}\,(Cosh(b\lambda s) - 1)\, F_{rn} + \right.$$

$$\left. + \frac{1}{\lambda^2}\left(\frac{1}{b\lambda} Sinh(b\lambda s) - s\right) F_{rc}\, F_{cn} \right] \lambda_n(0)$$

Type C

In this case we have $I_1 = 0$ and it is trivial to resolve the equation (18) :

$$\lambda_r(s) = \lambda_r(0) + \mu_r(0)s + \nu_r(0)\,\frac{s^2}{2}\,,$$

then (6) give us the worldline :

(22) $$x_r(s) = x_r(0) + \left(s\ \delta_{rn} + \frac{bs^2}{2}\, F_{rn} + \frac{b^2 s^3}{6}\, F_{rc}\, F_{cn} \right) \lambda_n(0)$$

Type D

$$\alpha = \pm\ i\beta b\,, \qquad \beta = \sqrt{\frac{I_1}{2}} > 0.$$

Here (18) has the solution

$$\lambda_r(s) = \lambda_r(0) + \frac{1}{b\beta}\, \mu_r(0) Sin(b\beta\, s) - \frac{1}{b^2\beta^2}\, \nu_r(0)\,(Cos(b\beta\, s) - 1).$$

where :

$$x_r(s) = x_r(0) + \Big[s\,\delta_{rn} - \frac{1}{b^2\beta^2}(Cos(b\beta s)-1)\,F_{rn} - \tag{23}$$

$$- \frac{1}{\beta^2}\left(\frac{1}{b\beta} Sin(b\beta s) - s\right) F_{rc}\,F_{cn} \Big]\,\lambda_n(0)$$

Now we must consider the type A, thus we shall study the right side of (17). The expressions (16) permit to write the Faraday tensor in function of its dual if into them we eliminate the Levi–Civita symbol, then :

$$*F_{rc} = \frac{\lambda}{\beta} F_{rc} - \frac{1}{\gamma_a \eta_a}\left(\beta + \frac{\lambda^2}{\beta}\right)(\gamma_r \eta_c - \gamma_c \eta_r)$$

which with (3) in (17) implies

$$\frac{d^2}{ds^2}\mu_r + b^2 H^2 Cos^2\frac{\gamma}{2}\mu_r = f_r(s) \tag{24}$$

where

$$f_r(s) = \frac{b^3\lambda\widetilde{\lambda}}{\gamma_a \eta_a}(\gamma_r \eta_c - \gamma_c \eta_r)\,\lambda_c. \tag{25}$$

However, from (3) and (15) it is easy to see that

$$\gamma_c \lambda_c = \lambda_c(0)\,\gamma_c\, e^{-b\lambda S},\ \ \eta_c \lambda_c = \lambda_c(0)\,\eta_c\, e^{b\lambda S}$$

which with (15) and (16) determine $f_r(s)$ via (25) :

$$f_r(s) = M_r\, Cosh(b\lambda s) + N_r\, Sinh(b\lambda s),$$

$$M_r = b^3\left(\lambda^2 F_{rc} - \frac{I_2}{4} *F_{rc}\right)\lambda_c(0), \tag{26}$$

$$N_r = b^3\lambda\left[\left(\widetilde{\lambda} + \frac{I_1}{4}\right)\delta_{rc} - F_{rn}\,F_{cn}\right]\lambda_c(0).$$

If we put (26) into (24) we obtain the solution :

$$\mu_r(s) = B_r Cos\left(bHs\,.\,Cos\frac{\gamma}{2}\right) + C_r Sin\left(bHs\,.\,Cos\frac{\gamma}{2}\right) + \frac{1}{2b^2\widetilde{\lambda}} f_r(s),$$

$$B_r = \mu_r(0) - \frac{1}{2b^2\widetilde{\lambda}} M_r, \tag{27}$$

$$C_r = \frac{1}{bHCos\frac{\gamma}{2}}\left(\nu_r(0) - \frac{\lambda}{2b\tilde{\lambda}} N_r\right),$$

with the corresponding trajectory :

$$x_r(s) = x_r(0) - \frac{1}{bHCos\frac{\gamma}{2}}\left[\left(Cos\left(bHs\,.\,Cos\frac{\gamma}{2}\right) - 1\right) B_r - \right.$$

$$\left. - Sin\left(bHs\,.\,Cos\frac{\gamma}{2}\right) C_r\right] + \frac{1}{2b^4\lambda^2\tilde{\lambda}}\left(f_r(s) - M_r\right) \tag{28}$$

due to the following identity :

$$\lambda_r(0) + \frac{1}{bHCos\frac{\gamma}{2}}\, C_r - \frac{1}{2b^3\lambda\tilde{\lambda}} N_r = 0 \tag{29}$$

with $\mu_r(0)$ and ν_r (0) given by (20).

Thus is completely known the motion of a point charge under the various types of constant electromagnetic fields. Our expressions (21), (22), (23) and (28) are equivalent to (4.6), (4.7), (4.8) and (4.9) of Piña [3].

REFERENCES

[1] J. Plebañski, *On algebraical properties of skew tensors*, Bull. Acad. Polon. Sci. Cl. 9 (1961) 587 and 593.

[2] J. L. Synge, *Timelike helices in flat spacetime*, Proc. R. Irish Acad. A 65 (1967) 27.

[3] E. Piña, *The Lorentz transformation and the motion of a charge into a constant electromagnetic field*, Rev. Mex. Fis. 16 (1967) 233.

[4] J. L. Synge, *Relativity. The special theory*, North Holland Pub ; Amsterdam (1965) Chap. 9.

[5] J. L. Synge, *Point-particles and energy tensors in special relativity*, Ann. Mat. Pura Appl. 84 (1970) 33.

[6] L. Infield and P.R. Wallace, *The equations of motion in electrodynamics*, Phys. Rev. 57 (1940) 797.

[7] J.L. Anderson, *Principles of relativity Physics*, Academic Press, N.Y. (1967)

[8] J. López–Bonilla, *Electrodynamics of classical charged particles*, Rev. Colomb. Fis. 17 (1985) 1

[9] E. Honig, E.L. Schucking and C.V. Vishveshwara, *Motion of charged particles in homogeneous electromagnetic fields*, J. Math Phys 15 (1974) 774

[10] J. López–Bonilla, G. Ovando and J. Rivera, *Intrinsic geometry of curves and the Bonnor's equation*, Proc. Indian Acad. Sci. (Math. Sci) 107 (1997) 43.

[11] J. López–Bonilla, G. Ovando and J. Rivera, *Lorentz-Dirac equation and Frenet-Serret formulae*, J. Moscow Phys. Soc. 9 (1999) 83.

[12] C. Lanczos, *Space through the ages*, Academic Press, London (1970).

[13] G.Y. Rainich, *Electrodynamics in the general relativity theory*. Trans. Am. Math. Soc. 27 (1925) 106.

[14] J. A. Wheeler, *Geometrodynamics*, Academic Press NY (1962)

[15] R. Penney, *Duality invariance and Riemannian geometry*, J. Math. Phys. 5(1964) 1431.

[16] J. L. Synge, *Principle null-directions defined in spacetime by an electromagnetic field*. Univ. Toronto Stud. Appl. Math. Ser. No 1 (1935)

[17] G. Ludwig, *Classification of electromagnetic and gravitational fields*, Am. J. Phys. 37 (1969) 1225

[18] K.P. Singh and S. R. Roy, *Classification of electromagnetic fields*, Indian J. Pure Appl. Math. 3 (1972) 532.

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