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# Lower Radical And Condition Q of Hemirings

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**Abstract:** In this paper, we geneeralize a few results of [7] for lower radical classes of rings for radical classes of hemirings, by using the Lee construction for lower radical classes of hemirings.

## 1. Introduction and Preliminaries :

D. M. Olson and T.L. Jenksins [6] discussed general Radical Theory of Hemirings. The theory was further enriched by many authors (see [3, 4, 9]). The certain condition (condition Q) was investigate by [7] for radical classes of rings. Here we are interesting to generalize a several results of [7] in the frame work of hemiring which is quite different from ring theoretical approach discussed in [7].

A semiring (R, +, .) is called a hemiring if (i) '+' is commutative (ii) there exists an element $0 \in R$ such that 0 is the identity of (R, +) and the zero element of (R, . ). i.e. $0r = r0 = 0,\ \forall\, r \in R$.

Lower radical classes for hemirings can be constructed similar to the construction of lower radicals for rings (see [8, 9]).

If R is a hemiring then $HR, D_1(R)$ denote the set of all homomorphic images of R and the set of all semi-ideals of R respectively. Observe that $\mathcal{M}$ is a homomorphically closed if and only if $HR \subseteq \mathcal{M},\ \forall\, R \in \mathcal{M}$ and ideally closed or hereditary if and only if $D_1(R) \subseteq \mathcal{M},\ \forall\, R \in \mathcal{M}$. If I is an semi-ideals of R, then we denote $I \leq R$.

First we include necessary preliminaries, let $\omega$ be the universal class of all hemirings and $\mathcal{M}$ be a sub-class of $\omega$ and let $\mathcal{M}_0$ be the homomorphic closure of $\mathcal{M}$ in $\omega$. For each $A \in \omega$, let $D_1(R)$ be the set of all semi-ideals of R. Inductively we define.

$$D_{n+1}(R) = \{I : I \text{ is an semi-ideal of some hemiring in } D_n(R)\}.$$

Let $D(R) = \bigcup_{n\in N} D_n(R)$, n = 1, 2, 3, . . . By using ring theoretical approach discussed in [7], (see also [5, 10]), we have

$£\mathcal{M} = \{R \in \omega : D(R/I) \cap \mathcal{M}_0 \neq 0$, for each proper semi-ideal I of R$\}$, is the Lee construction for lower radical determined by $\mathcal{M}$, and $\mathcal{M} \subseteq £\mathcal{M}$, (see also [6, 9, 10]). For undefined terms of hemirings we may refer (see [1, 2, 6]).

## 2. Condition Q:

We extend the result of [7] by using the above Lee construction of lower radical for hemiring which is indeed provides an excellent and different approach to handle the many results of [7] in the frame work of hemiring.

If ρ is a radical class then its semisimple class is denoted by Sρ. The following definition is closely inspired by [7].

**Definition 2.1 :** Let $\mathcal{M}$ be an arbitrary class of hemirings. A non simple hemiring R is called **$H_M$ - hemiring** if

i) $R / I \in \mathcal{M}$, for every $(I \neq 0) \leq R$

ii) Every minimal semi-ideal J or R belongs to $\mathcal{M}$.

A radical class ρ satisfies the **condition Q** for $\mathcal{M}$ if $H_M \subseteq \rho \cup S\rho$. We assume that $0 \in H_M$. Observe that a non-zero simple hemiring $R \in \mathcal{M}$ if and only if $R \in H_M$.

If R is a simple hemiring, $R \in \mathcal{M}$, $R / R = 0 \in \mathcal{M}$, i.e. (i) is satisfied. Moreover R is minimal semi-ideal of R, i.e. $R \in \mathcal{M}$, thus (ii) is satisfied and hence $R \in H_M$. Conversely assume that $R \in H_M$. Since (i), (ii) are satisfied. Also R is minimal semi-ideal of R, therefore by (ii) $R \in \mathcal{M}$.

The following theorem was proved by H. J. le Roux and G.A.P Heyman [7] for rings. Here we generalize it for hemiring, which can be obtained on the lines of ring theoretical approach.

**Theorem 2.2.** *Let $\mathcal{M}$ be a homomorphically, ideally closed class of hemirings, which is further closed under finite direct sum. Let ρ be a radical class such that $\rho \cap H_M \neq 0$. Then $H_M \subseteq \rho \cup S\rho$ if and only if $\mathcal{M} \subseteq \rho$.*

**Theorem 2.3 :** *Let $\mathcal{M}$ be a hereditary class which is homomorphically closed and closed under the finite direct sums respectively. A radical class $\rho = £\mathcal{M}$ if and only if*

i) ρ satisfies the condition Q i.e. $H_M \subseteq \rho \cup S\rho$

ii) $\rho \cap H_M \neq 0$

iii) for any radical class ς satisfies (i), (ii) implies that $\rho \subseteq \zeta$..

**Definition 2.4:** *A hemiring R is subdirectly irreducible if and only if the intersection of any collection of non-zero k-ideals of A is again a non-zero k-ideal.*

Indeed intersection of all non-zero k-ideal of subdirectly irreducible hemiring is non-zero k-ideal which is uniquely determined and is called heart of R and is denoted by H.

**Theorem 2.5.** *If ρ is hereditary radical class and A is subdirectly irreducible with heart H then*

i) $H \in \rho \cup S\rho$

ii) $H \in S\rho$ if and only if $A \in S\rho$.

iii) Then H is either simple hemiring or a zero-hemiring.

**Proof:**

i) By [ 6, Theorem 3], ρ(H) is k-ideal of H. By definition of heart, we have $\rho(H) = H$ or $\rho(H) = 0$. This implies that $H \in \rho$ or $H \in S\rho$ and hence $H \in \rho \cup S\rho$.

ii) Let $A \in S\rho$. As $H \leq A$, by hereditary of Sρ [9, Theorem 1] we have $H \in S\rho$.

Conversely let $H \in S\rho$. This implies that $\rho(H) = 0$. Since ρ is hereditary radical class. So by [6, Lemma 8] we have $\rho(H) = H \cap \rho(A)$. Thus $\rho(H) = 0$ and hence $H \cap \rho(A) = 0$. Now, $H \cap \rho(A) = 0$. We claim that $\rho(A) = 0$. If $\rho(A) \neq 0$. As $H \neq 0$ then $\{H, \rho(A)\}$ is family of non zero semi-ideal of A such that $H \cap \rho(A) = 0$. This implies that A is not subdirectly irreducible, which is a contradiction. Hence $\rho(A) = 0$. This implies that $A \in S\rho$.

iii) As H is minimal semi-ideal, $H^2 \leq H$, therefore, we have $H^2 = \{0\}$ or $H^2 = H$. If $H^2 = \{0\}$ then H is zero hemiring. If $H^2 = H$, let $(I \neq 0)$ be non-zero semi-ideal of H. Assume that

$$I^* = I + AI + IA + AIA$$

then $(0 \neq I^*)$ is k-ideal of A, generated by I. Moreover $(I^*)^3 \subseteq I$ (see [6], Lemma 9, p. 28). By $I \subseteq H$ and $H \leq A$, we have $AI \subseteq AH \subseteq H$, $IA \subseteq HA \subseteq H$, $AIA \subseteq AHA \subseteq H$. This implies that $I^* = I + AI + IA + AIA \subseteq H$ i.e. $I^* \subseteq H$. By definition of H, we have $I^* = H$ or $I^* = 0$. As $I \neq 0$, therefore $I^* \neq 0$ ($\because I \subseteq I^*$) and hence $I^* = H$. This implies that $(I^*)^3 = H^3$. As $(I^*)^3 \subseteq I$, this implies that $H^3 = I$ or $H^2 H \subseteq I$ and hence $H^2 \subseteq I$ ($\because H^2 = H$). Consequently we have $H \subseteq I$. This implies that $H = I$ and hence H is a simple hemiring.

First we note that Z = class of all zero hemirings and $\beta_s$ = upper radical determined by the class of all prime simple hemirings.

The following theorem provides a necessary and sufficient condition for given radical class ρ of hemiring to satisfy the condition Q associated to certain class of hemirings, which is indeed an extension of Theorem 2 of [7].

**Theorem 2.6.** *Let $\mathcal{M}$ be a hereditary class which is homomorphically closed and also closed under the finite direct sum of hemirings and $Z \subseteq \mathcal{M}$. A hereditary radical class $0 \neq \rho \subseteq \beta_s$ satisfies condition* Q *if and only if $\mathcal{M} \subseteq \rho$.*

**Proof:** If $\mathcal{M} \subseteq \rho$. *then $M \cap \rho = M \neq 0$, and hence $H_M \cap \rho \neq 0$. By Theorem 2.2,* ρ *satisfies condition* (*Q*).

Conversely assume that ρ satisfies condition (Q). Let us suppose $\mathcal{M} \not\subset \rho$. Now there are two cases :

**CASE I :** Let $\mathcal{M} \cap \rho \neq 0$, as $\mathcal{M} \subseteq H_M$. This implies that $H_M \cap \rho \neq 0$. As $\mathcal{M} \not\subset \rho$, by Theorem 2.2, ρ does not satisfy (Q), a contradiction to the assumption.
Hence $\mathcal{M} \subseteq \rho$.

**CASE II :** Let $\mathcal{M} \cap \rho = 0$, we will show that, this is impossible.
Let $(0 \neq R) \in \rho$. As R can be decomposed into sub-direct sum of subdirectly irreducible hemirings $R_i$, s.

Let $R_i$ be an arbitrary component of R, and let H be heart of R, then $H^2 = H$ or H= 0. If $H^2 = H$, then H is a simple prime hemiring. Since $\beta_s$= upper radical determined by all simple prime hemirings, therefore $H \in S\beta_s$. Since $\rho \subseteq \beta_s$, therefore $S\beta_s \subseteq S\rho$ (see [6]) and hence $H \in S\rho$. By Theorem 2.5, $H \in S\rho$ if and only if $R_i \in S\rho$.

As R is the direct sum of $R_i$, therefore there exists is a projection mapping $\pi_i : R \to R_i$. If ker $\pi_i = I_i$ then $R / I_i \cong R_i$. Since $R \in \rho$, and ρ is a radical class, therefore $R_i \in \rho$, ∀ i. Since $R_i \in S\rho$, therefore $R_i \in S\rho \cap \rho$ (See [10]). This means $R_i = 0$. As $R_i$ is an arbitrary component of R such that $R_i = 0$, therefore R = 0, a contradiction. Hence $H^2 \neq H$, the minimality of H implies $H^2 = 0$. i.e. H is zero-hemiring i.e. $H \in Z \subseteq \mathcal{M}$. This implies that $H \in \mathcal{M}$. Since $R \in \rho$, and $H \leq R$ and ρ is hereditary. This implies that $H \in \rho$, this implies that $H \in \rho \cap \mathcal{M} = 0$. Consequently, we have H = 0, which contradicts the definition of heart H. Hence $\rho \cap \mathcal{M} = 0$ is impossible i.e. Thus $\rho \cap \mathcal{M} \neq 0$, so result follows by case (I).

**Theorem 2.7:** *A hereditary radical class* ρ *under the hypothesis of the Theorem 2.6 coincides with £$\mathcal{M}$ if and only if*

i) ρ satisfies condition Q
ii) for only hereditary class ς satisfying condition Q then $\rho \subseteq \varsigma$.

**Proof :** Let £$\mathcal{M}$ = ρ. By Theorem 2.3, ρ satisfies condition Q and $\rho \cap H_M \neq 0$. Let ς be a radical class such that ς satisfies condition Q, then by Theorem 2.6, $\mathcal{M} \subseteq \varsigma$. This implies that $\mathcal{M} \cap \varsigma \neq 0$. Thus $H_M \cap \varsigma \neq 0$. Thus ς satisfies (i), (ii), of Theorem 2.3, therefore by (iii) of Theorem 2.3, $\rho \subseteq \varsigma$. Conversely assume that (i), (ii) of the statement are valid, ρ satisfies condition Q, therefore by Theorem 2.6, $\mathcal{M} \subseteq \rho$. This implies that $\pounds\mathcal{M} \subseteq \rho$. Now $\mathcal{M} \subseteq \pounds\mathcal{M}$. This implies that $\pounds\mathcal{M}$ satisfies condition Q (by Theorem 2.6). By (ii) of the statement $\rho \subseteq \pounds\mathcal{M}$. This implies that $\pounds\mathcal{M} = \rho$.

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