The Nepali Math.Sc. Report
Vol. 19. No. 1 & 2, 2001

On a structure defined by a tensor field $f(\neq 0)$ of type (1,1) satisfying $(f^{2v+3}+\lambda^2 f)\,(f^{2v+3}+\mu^2 f)=0$

MOHD. NAZRUL ISLAM KHAN

Abstract: Androu [1] has studied the structure defined by a tensor field $f\,(\neq 0)$ of type (1,1) satisfying $f^5+f=0$. In the present paper, we have defined and studied $f_{\lambda\mu}(2v+3,1)$–structure. We have also obtained a positive definite Riemanian metric with respect to which the complementary distributions are orthogonal.

1. Introduction

$f_{\lambda,\mu}(2v+3,1)$–Structure

Let M^n be an n-dimensional differentiable manifold of class C^∞. Suppose there exists on M^n a non-null tensor field f of type (1.1) of class C^∞ and of rank r satisfying.

(1.1) $$(f^{2v+3}+\lambda^2 f)\,(f^{2v+3}+\mu^2 f)=0, \quad \lambda\neq\mu$$

Let us defined on such M^n tensor fields 'ℓ' and 'm' of type (1.1) as follows

(1.2) $$\ell=\frac{f^{2v+2}+\lambda^2}{\lambda^2-\mu^2} \quad m=\frac{f^{2v+2}+\lambda^2}{\mu^2-\lambda^2}$$

Then it can be easily shown that

(1.3) $$\ell^2=\ell, \quad m^2=m, \quad \ell m=\ell m=0, \quad \ell+m=1$$

Thus the operators ℓ and m when applied to tangent space of M^n at a point all complementary projection operators. Thus there exist complementary distributions L and M corresponding to projection operators ℓ and m respectively. Let us call such structure as $f_{\lambda,\mu}(2v+3,1)$–structure.

For the manifold M^n equipped with $f_{\lambda,\mu}(2v+3,1)$–structure, the following result can be proved easily.

(i) $$f\ell=\frac{f^{2v+2}+f\lambda^2}{\lambda^2-\mu^2}, \quad fm=\frac{f^{2v+2}+f\mu^2}{\mu^2-\lambda^2}$$

(1.4) (ii) $$f^2\ell = -\mu^2\ell \ , \quad f^2 m = -\lambda^2 m$$

and

(iii) $$m - \ell = \frac{2f^{2\nu+2} + \mu^2 + \lambda^2}{\mu^2 - \lambda^2}$$

2. $f_{\lambda,\mu}(2\nu+3,1)$–structure in local Coordinates:

We now introduce in the manifold M^n a local coordinate system and represented by f_i^h, ℓ_i^h and m_i^h the local components of f, ℓ and m respectively. We also introduce in M^n, a positive definite Riemannian metric by taking r mutually orthogonal unit vectors u_a^h in $L(a, b, c \ldots\ldots = 1,2, \ldots\ldots r)$ and $n - r$ mutually orthogonal unit vectors u_B^h in $L(A, B, C, \ldots = 1,2, \ldots\ldots n-r)$ in M. Thus we have

(2.1) $$\ell_i^h, u_b^i = u_b^h, \quad \ell_i^h, u_B^i = 0 \ ;$$
$$m_i^h, u_b^i = 0, \quad m_i^h, u_B^i = u_B^h,$$

Let (v_i^a, v_i^A) be the matrix inverse to (u_b^h, u_B^h). Then v_i^a and v_i^A are components of linearly independent covariant vectors satisfying.

(2.2) $$v_i^a, u_b^i = \delta_b^a, \quad v_i^a \ u_B^i = 0 \ ;$$
$$v_i^A, u_b^i = 0, \quad v_i^A, u_B^i = \delta_B^A,$$

δ_j^i being Kronecker delta. Also

(2.3) $$v_i^a, u_a^h + v_i^A \ u_A^h = \delta_i^h$$

In view of equations (2.1) and (2.2), we have

(2.4) $$(\ell_i^h, v_h^a) \ u_b^i = \delta_b^a, \quad (\ell_i^h, v_h^a) \ u_B^i = 0$$
$$(m_i^h, v_h^A) \ u_b^i = 0, \quad (m_i^h, v_h^A) \ u_B^i = \delta_B^A$$

Thus we have

(2.5) $$\ell_i^h, v_h^a = v_i^a, \quad \ell_i^h, v_h^A = 0 \ ;$$
$$m_i^h, v_h^a = 0, \quad m_i^h, v_h^A = v_i^A,$$

Since $fm = 0$, we have $f_i^h \ m_h^i = 0$. Hence contracting with v_j^A and making use of (2.5), we obtain

(2.6) $$f_i^h, v_h^A = 0$$

further, since

$$\ell_j^h, u_a^j = u_a^h,$$

we have $$\ell_j^h, u_a^j \, v_i^a = v_i^a, u_a^h$$

or $$\ell_i^h, (\delta_i^j - v_i^A \ u_A^j) = \ v_i^a, u_a^h$$

(2.7) $$\ell_i^h = v_h^a\, u_b^i$$

by virtue of (2.1) and (2.3). Similarly

(2.8) $$m_i^h = v_i^A\, u_A^h$$

Let us now put

(2.9) $$g_{ji} = v_j^a\, v_i^a + v_j^A\, v_i^A$$

Then g_{ji} is globally defined positive definite Riemannian metric with respect to which (u_b^h, u_B^h) form an orthogonal frame and such that

(2.10) $$v_j^a = g_{ji}\, u_a^i, \qquad v_j^A = g_{ji}\, u_A^i$$

If we further put

(2.11) $$\ell_{ji} = \ell_j^t\, g_{ti} \text{ and } m_{ji} = m_j^t\, g_{ti},$$

We have in view of (2.7) and (2.8)

(2.12) $$\ell_{ji} = v_j^a\, v_i^a, \qquad m_{ji} = v_j^A\, v_i^A$$

and consequently

(2.13) $$\ell_{ji} + m_{ji} = g_{ji}$$

The following equations can be proved easily

(2.14)
(i) $\ell_j^t\, \ell_i^s\, g_{ts} = \ell_{ji}$,
(ii) $\ell_j^t\, m_i^s\, g_{ts} = 0$
and (iii) $m_j^t\, m_i^s\, g_{ts} = m_{ji}$

For any two vectors x, y with components x^i, y^i, let us put

(2.15) $$m^*(x,y) = m_{st}\, x^s\, y^t, \qquad g(x,y) = g_{st}\, x^s y^t$$

(2.16) $$G(x,y) = \frac{1}{2(v+1)}\{g(x,y) + g(fx, fy) + + g(f^2x, f^2y) + \dots + g(f^{2v+1}x, f^{2v+1}y) + m^*(x,y)\}.$$

Thus we have

$$m^*(u_A, u_a) = g(u_A, u_a) = g(fu_A, fu_a) = = g(f^2u_A, f^2u_a) = \dots = g(f^{2v+1}u_A, f^{2v+1}u_a) = 0$$

$$G(u_A, u_a) = \frac{1}{2(v+1)}\{g(u_A, u_a) + g(fu_A, fu_a) + + g(f^2u_A, f^2u_a) + \dots + g(f^{2v+1}u_A, f^{2v+1}u_a) + m^*(u_A, u_a) = 0.$$

By virtue of the fact that the distributions L and M are orthogonal with respect to Riemannian metric g. Thus L and M are orthogonal with respect to G also. Consequently, we have the following theorem.

Theorem 2.1. *Let M^n be an n-dimensional differentiable manifold equipped with $f_{\lambda,\mu}(2v+3,1)$-Structure of rank r. Then there exist complementary distributions L and M and a positive definite Riemannian metric G with respect to which the distributions are orthogonal.*

REFERENCES

[1] Androu, F. Gouli. *On a structure defined by a tensor field f satisfying* $f^5+f=0$, Tensor, N.S., Vol. 29 (1982), pp. 249–254.

[2] Mishra, R.S. *On almost product and decomposable manifolds*, Tensor, N.S., Vol. 21(1970), pp. 255–260.

[3] Nivas, Ram and Prasad, C.S. *On a structure defined by a tensor field f of type* (1.1) *satisfying* $f^{2k}+f^2=0$. To appear, Analete Universitatea din Timi, Soara, Romania.

MOHD. NAZRUL ISLAM KHAN
Department of Mathematics
Amiruddaula Islamia Degree College,
Lalbagh, Lucknow-226001
(U.P.) India.