The Nepali Math. Sc. Report
Vol. 27. No.1 & 2. 2007.

On *APST* Riemannian manifold with second order generalised structure

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In this present paper, we have studied some properties of a differentiable manifold and also studied the Almost para sasakian type (APST)-Riemannian manifold.

Introduction :

Let an n-dimensional Riemannian manifold M_n, on which there are defined a tensor field F of type (1,1) a tensor field T, a 1-form A and metric tensor g satisfying for arbitrary vector field X, Y, Z and a is any complex number (non-zero).

(1.1) $$F^2 X = a^2 X - A(X)T$$

(1.2) $$\bar{X} = F(X)$$

(1.3) $$A(T) = -a^2$$

(1.4) $$A(FX) = 0$$

(1.5) $$F(T) = 0$$

(1.6) $$g(T, X) = A(X)$$

(1.7) $$g(FX, FY) = -a^2 g(X,Y) + A(X) A(Y)$$

then structure (F, T, A, g) is called almost para contact metric structure and manifold M_n will be called Almost para contact metric Riemannian manifold.

Let us call such a structure a generalized almost contact metric structure

Let us define

(1.8) $${}'F(\bar{X},Y)=g(FX,Y)$$

And barring X in (1.8) we have

(1.9) $${}'F(\bar{X},Y)=g(F^2X,Y)$$

Which by virtue of (1.1) yields

(1.10) $${}'F(X,Y)=a^2g(X,Y)-A(X)A(Y)$$

Now barring Y in (1.8) we have

(1.11) $${}'F(X,\bar{Y})=g(FX,FY)$$

This with the help of (1.7) becomes

(1.12) $${}'F(X,\bar{Y})=-\{a^2g(X,Y)-A(X)A(Y)\}$$

Thus from the relation (1.10) and (1.12) we have

(1.13) $${}'F(X,\bar{Y})={}'F(\bar{X},Y)$$

Replacing X by T in equation (1.18) and making use of (1.5), we obtain-

(1.14) $${}'F(T,Y)=0$$

Barring X in equation (1.12) and making use of (1.1) and (1.14) we get

(1.15) $${}'F(\bar{X},\bar{Y})=-a^2F(X,Y)$$

Now barring Y in (1.7) and making use of (1.4) and (1.5) in the resulting equation, we obtain

(1.16) $$g(FX,Y)=-g(X,FY)$$

Thus from the equation (1.8) and (1.16), we have

(1.17) $${}'F(FX,Y)=-{}'F(Y,X)$$

2, Nijenhuis Tensor

Nijenhuis Tensor is given by

(2.1) $$N(X,Y)=[\bar{X},\bar{Y}]+\overline{[X,Y]}-\overline{[\bar{X},Y]}-\overline{[X,\bar{Y}]}$$

Making use of (1.1) in (2.1), we get-

(2.2) $$N(X,Y)=[\bar{X},\bar{Y}]+a^2[X,Y]-A([X,Y])T-\overline{[\bar{X},Y]}-\overline{[X,\bar{Y}]}$$

Now let us put

(2.3) $$P(X,Y)=[\bar{X},\bar{Y}]-\overline{[X,\bar{Y}]}$$

(2.4) $$Q(X,Y)=[\bar{X},\bar{Y}]-\overline{[\bar{X},Y]}$$

(2.5) $$H(X,Y)=[\bar{X},\bar{Y}]+a^2[X,Y]$$

Theorem (2.1) *The Nijenhuis tensor and P(X, Y) are related as-*

(2.6) $$a^2 P(X,Y) - \overline{P(X,\bar{\bar{Y}})} = a^2 N(X,Y) - A(Y) - \overline{[\bar{\bar{X}},T]} + a^2 A(Y)[X,T] + A(Y)A([X,T])T$$

Proof : Barring Y in (2.3) and using (1.1), we obtain-

(2.7) $$P(X,\bar{Y}) = a^2[\bar{X},Y] + A(Y)[\bar{X},T] - a^2\,\overline{X,Y]} - \overline{A(Y)[X,T]}$$

Again barring the above equation and making use of (1.1)

$$\overline{P(X,\bar{Y})} = a^2\overline{[\bar{X},Y]} + A(Y)\overline{[\bar{X},T]} - a^2\{a^2[X,Y] - A[X,Y]T\} - \{a^2 A(Y)[X,T] - A(Y)A([X,T])T\}$$

(2.8) $$P[\bar{X},\bar{Y}] = a^2\overline{[\bar{X},Y]} + A(Y)\overline{[\bar{X},T]} - a^4[X,Y] + a^2[X,T]T - a^2 A(Y)[X,T] + A(Y)A([X,T])T$$

Now from the equation (2.3) and (2,8), we obtain

(2.9) $$a^2 P(X,Y) - \overline{P(X,\bar{Y})} = a^2[\bar{X},\bar{Y}] - a^2\overline{[\bar{X},Y]} + a^4[X,Y] - A(Y)\overline{[\bar{X},T]} - a^2 A([X,Y])T + a^2 A(Y)[X,T] - A(Y)A([X,T])T$$

Making use of (2.2) in (2.9) we get the result.

Corollary (2.1): *In a differentiable manifold* M^n.

We have

(2.10) $$a^2 P[X,T] = a^2 N(X,T) + a^2\overline{[\bar{X},T]} - a^4[X,T] - a^2 A([X,T])T$$

Proof: Putting T for Y in (2.6) and using (1.5)and (1.3), we get the result.

Theorem (2.2): *In a differentiable manifold* M^n,

Weh have

(2.11) $$a^2 Q(X,Y) - Q\overline{(\bar{\bar{X}},Y)} = a^2 N(X,Y) - A(X)\overline{[T,\bar{\bar{Y}}]} + a^4 A(X)[T,Y] + A(X)A([T,Y])T$$

Proof: Barring X in (2.4) and making use of (1.1), we get

(2.12) $$Q(\bar{X},Y) = a^2[X,\bar{Y}] + A(X)[T,\bar{Y}] - a^2\overline{[\bar{X},Y]} + A(X)\overline{[T,Y]}$$

Now barring the whole equation (2.12) and making use of (1.1), we get-

$$\overline{Q(\bar{X},Y]} = a^2\overline{[X,\bar{Y}]} + A(X)\overline{[T,\bar{Y}]} - a^2\{a^2[X,Y] - A([X,Y])T\} + A(X)\{a^2(T,Y] - A([T,Y)T\}$$

(2.13) $$\overline{Q(\bar{X},Y]} = a^2\overline{[X,\bar{Y}]} + A(X)\overline{[T,\bar{Y}]} - a^4[X,Y] + a^2 A([X,Y])T + a^2 A(X)[T,Y] - A(X)A([T,Y)T$$

Now from (2.4) and (2.13) we get

$$(2.14) \qquad a^2 Q(X,Y) - Q[\overline{\bar{X},Y}] = a^2[\bar{X},\bar{Y}] - a^2[\overline{X,\bar{Y}}] - a^2[\overline{\bar{X},Y}] + a^4[X,Y]$$
$$-a^2 A([X,Y])T - A(X)[\overline{T,\bar{Y}}] - a^4 A(X)[T,Y] - A(X)\,A([T,Y])T.$$

Thus from (2.2) and (2.14) we obtain the required result.

Corollary (2.2): *In a generalized almost contact metric manifolds M^n we have*

$$(2.15) \qquad a^2 Q(T,Y) = a^2 N(T,Y) + a^4[\overline{T,\bar{Y}}] - a^4[T,Y] = -a^2 A([T,Y])T$$

Proof: Replacing X by T in (2.11) and using (1.3) and (1.5), we get the equation (2.15)

Theorem (2.3). *In a generalized almost contact metric structure manifold M^n*

$$(2.16) \qquad a^2 H(X,Y) - H[\overline{\bar{X},Y}] = a^2 N[X,Y] - a^2 A([X,Y])T - A(X)[\overline{T,\bar{Y}}]$$

Proof: Barring X in (2.5) and making use of (1.1)

$$(2.17) \qquad H(\bar{X},Y) = a^2[X,\bar{Y}] + A(X)[T,\bar{Y}] - a^2(\bar{X},Y)$$

Now barring the whole equation (2.17) and making use of (1.1)

$$(2.18) \qquad H[\overline{\bar{X},Y}] = a^2[\overline{X,\bar{Y}}] - A(X)[\overline{T,\bar{Y}}] + a^2[\overline{\bar{X},Y}]$$

Thus with the help of (2.2), (2.5) and (2.18) we get (2.16)

Corollary (2.3): *The equation* (2.16) *is equivalent to*

$$(2.19) \qquad a^2 H(T,Y) = a^2 N[T,Y] - a^2 A([T,Y]) + a^2[\overline{T,\bar{Y}}]$$

Proof: Replacing X by T in (2.16) and using the equation (1.3) and (1.5), we get the result.

Theorem (2.4): *In a generalized almost contact metric structure manifold M^n, we have*

$$(2.20) \qquad H(T,Y) - Q[T,Y] = a^2[T,Y]$$

Proof: Equation (2.20) follows directly wih the help of equation (2.15) and (2.19)

Theorem (2.5): *In a generalized almost contact metric manifold M^n, we have*

$$(2.21) \qquad a^2 H(X,Y) - H[\overline{\bar{X},Y}] = \{a^2 P[X,Y] - P[\overline{X,\bar{Y}}] + A(Y)[\overline{\bar{X},T}] - a^2 A(Y)[X,T]$$
$$-A(Y)\,A[(X,T)T\} - a^2 T([X,Y])T - A(X))[\overline{T,Y}]$$

Proof: Proof follows with the help of equation (2.6) and (2.16)

Theorem (2.6): *In order that a generalized almost contact metric manifold by completely integrable it is necessary that*

(2.22) $$A[\bar{X},\bar{Y}]T = 0$$

Proof: Barring X in (2.2) and with the help of equation (1.1), we get

(2.23) $$N(\bar{X},Y) = a^2(X,\bar{Y}) - A(X)[T,\bar{Y}] + a^2[\bar{X},Y] - A([\bar{X},Y])T \\ -\overline{[\bar{X},Y]} - a^2\,\overline{[X,Y]} + A(X)\overline{[T,Y]}$$

Now barring the whole equation and using (1.1), we obtain

(2.24) $$N\overline{[\bar{X},Y]} = a^2\overline{[X,\bar{Y}]} - A(X)\overline{[T,\bar{Y}]} + a^2\overline{[\bar{X},Y]} - a^2[\bar{X},\bar{Y}] + A([\bar{X},\bar{Y}])T - a^4[X,Y] \\ + a^2A([X,Y])T + a^2A(X)[T,Y] + A(X)\,A([T,Y])T$$

From the equation (2.2) and (2.24), we have

(2.25) $$N\overline{[\bar{X},Y]} + a^2N(X,Y) = -A(X)\,(\overline{[T,\bar{Y}]}) + A[\bar{X},\bar{Y}]T + a^2A(X)\,[T,Y] \\ + A(X)A([T,Y])T$$

(2.26) $$N(T,Y) = a^2[T,Y] + A([T,Y])T - \overline{[T,\bar{Y}]}$$

Using (2.26) in (2.25) we obtain

(2.27) $$N\overline{[\bar{X},Y]} + a^2N(X,Y) = A(X)N(T,Y) + A([\bar{X},\bar{Y}])T$$

For completely integrable manifold equation (2.27) reduces to equation (2.22)

Theorem (2.7) *In a completely integrable generalized almost contact metric structure manifold* M^n, *we have the following result.*

(2.28) $$A(X)\{[T,\bar{Y}] - \overline{[T,Y]}\} + A([\bar{X},Y])T = A(Y\{[\bar{X},T] - \overline{[X,\bar{T}]} + A([X,\bar{Y}])T$$

Proof: Barring X in equation (2.2) and making use of (1.1), we get

(2.29) $$N(\bar{X},Y) = a^2(X,\bar{Y}) - A(X)[T,\bar{Y}] + a^2[\bar{X},Y] - A([\bar{X},Y])T - \overline{[\bar{X},Y]} \\ - a^2\overline{[X,Y]} + A(X)\,\overline{[T,Y]}$$

Again barring Y in equation (2.2) and making use of (1.1), we get

(2.30) $$N(X,\bar{Y}) = a^2[\bar{X},Y] - A(Y)[\bar{X},T] + a^2[X,\bar{Y}] + A([X,\bar{Y}])T \\ - a^2\overline{[X,Y]} + A(Y))\,\overline{[X,T]}$$

Now from these two equation (2.29) and (2.30) and using $N(X, Y)$, we have the required result (2.28)

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