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On Complete and Horizontal Lifts from a Manifold with HSU–(4,2) Structure to its Cotangent Bundle

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Abstract : Manifolds with $f(4,2)$-structure have been defined and studied by Yano, Houh and Chen [3] and others. Geometry of tangent and cotangent bundles in a differentiable manifold has been studied by Yano and Ishihara [4]. Hsu-structure had been defined by Prof. Mishra [2]. The purpose of this chapter is to study complete and horizontal lifts from a manifold with Hsu-(4,2) structure to its cotangent bundle.

1. Preliminaries

Let M be a differentiable manifold of class C^∞ and dimension n and let C_{TM} denote the cotangent bundle of M. Then C_{TM} is also a differentiable manifold of class C^∞ and dimension $2n$ [4]. Throughout this chapter, the following notations and conventions will be used :

(i) The map $\pi : C_{TM} \rightarrow M$ is the projection map of C_{TM} onto M.

(ii) Suffixes $a, b, c, \ldots i, j, \ldots$ Take the value 1 to n and $\bar{i} = i + n$. Suffixes $A, B, C, \ldots$ take the values 1 to $2n$

(iii) $\mathfrak{J}_s^r(M)$ denotes the set of tensor fields of class C^∞ and of type (r,s) on M. Similarly $\mathfrak{J}_s^r(C_{TM})$ denotes the set of tensor fields of class C^∞ and of type (r, s) in C_{TM}.

(iv) Vectors in M are denoted by $X,Y,Z, \ldots$ and the Lie derivative by $\mathcal{L}_X$. The Lie product of X,Y is denoted by $[X,Y]$.

If A is a point in $M, \pi^{-1}(A)$ is a fibre over A. Any point $p \in \pi^{-1}(A)$ is the ordered pair (A, p_A) where p is 1-form in M and p_A is the value of p at A. Let U be a coordinate neighbourhood in M such that $A \in U$. Then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in C_{TM} and $p \in \pi^{-1}(U)$.

2. Complete Lift of *HSU*–(4,2) Structure

Let M be an n-dimensional differentiable manifold of class C^∞. Suppose there exist on M a tensor field $f(\neq 0)$ of type (1,1) satisfying

(2.1) $$f^4 - \lambda^r f^2 = 0$$

where λ is complex number not equal to zero and r some finite integer. In such a manifold M, let us put

(2.2) $$\lambda = f^2/\lambda^r \text{ and } m = I - f^2/\lambda^r$$

where I denotes the unit tensor field. Then it is easy to show

(2.3) $$\lambda^2 = \lambda,\ m^2 = m,\ \lambda + m = I,\ \lambda m = m\lambda = 0.$$

Thus the operators λ and m when applied to the tangent space of M at a point are complementary projection operators. Hence there exist complementary distributions L^* and M^* corresponding to the projection operators λ and m respectively. If the rank of 'f' is constant everywhere and equal to r, the dimensions of L^* and M^* are r and $(n-r)$ respectively. Let us call such a structure as Hsu-(4,2) structure of rank r.

Let f_i^h be the component of f at A in the coordinate neighbourhood U of M. Then the complete lift f^C of f is also a tensor field of type (1,1) in C_{TM} whose components $\tilde{f}_B^A$ in $\pi^{-1}(U)$ are given by

(2.4.1) $$\tilde{f}_i^h = f_i^h$$

(2.4.2) $$\tilde{f}_{\bar{i}}^h = 0$$

(2.4.3) $$\tilde{f}_i^{\bar{h}} = p_a[\partial f_h^a / \partial x^i - \partial f_i^a / \partial x^h]$$

and

(2.4.4) $$\tilde{f}_{\bar{i}}^{\bar{h}} = f_h^i$$

where $(x^1, x^2, x^3, \ldots, x^n)$ are coordianates of A in U and p_A has components $(p_1, p_2, p_3, \ldots, p_n)$. Thus we can write

(2.5) $$f^C = (\tilde{f}^A_B) = \begin{bmatrix} f^h_i & 0 \\ p_a(\partial_i f^a_h - \partial_h f^a_i) & f^i_h \end{bmatrix}$$

where $\partial_i = \partial / \partial x^i$.

If we put

(2.6) $$\partial_i f^a_h - \partial_h f^u_i = 2\partial\,[if^a_h].$$

then the equation (2.5) can be written as

(2.7) $$f^C = (\tilde{f}^A_B) = \begin{bmatrix} f^h_i & 0 \\ 2p_a \partial[if^a_h] & f^i_h \end{bmatrix}$$

$$\Rightarrow \qquad (f^C)^2 = \begin{bmatrix} f^h_i & 0 \\ 2p_a \partial[if^a_h] & f^i_h \end{bmatrix} \begin{bmatrix} f^i_j & 0 \\ 2p_t \partial[jf^t_i] & f^j_i \end{bmatrix}$$

$$= \begin{bmatrix} f^h_i f^i_j & 0 \\ 2p_a f^i_j \partial[if^a_h] + 2p_t f^i_h \partial[jf^t_i] & f^i_h f^j_i \end{bmatrix}$$

If we substitute

(2.8) $$2p_a f^i_j \partial[if^a_h] + 2p_t f^i_h \partial[jf^t_i] = L_{hj}$$

then we can write

(2.9) $$(f^C)^2 = \begin{bmatrix} f^h_i f^i_j & 0 \\ L_{hj} & f^j_i f^i_h \end{bmatrix}$$

Squaring (2.9) again we get

(2.10) $$(f^C)^4 = \begin{bmatrix} f^h_i f^i_j & 0 \\ L_{hj} & f^j_i f^i_h \end{bmatrix} \begin{bmatrix} f^j_k f^k_\lambda & 0 \\ L_{j\lambda} & f^\lambda_k f^k_j \end{bmatrix}$$

$$= \begin{bmatrix} f^h_i f^i_j f^j_k f^k_\lambda & 0 \\ f^j_k f^k_\lambda L_{hj} + f^j_i f^i_h L_{j\lambda} & f^\lambda_k f^k_j f^j_i f^i_h \end{bmatrix}$$

Putting

(2.11) $$f^j_k f^k_\lambda L_{hj} + f^j_i f^i_h L_{j\lambda} = \lambda^r L_{hj}$$

Then in view of equation (2.11) and (2.1) the equation (2.10) take the form

$$(f^C)^4 = \begin{bmatrix} \lambda^r f_t^h f_\lambda^t & 0 \\ \lambda^r L_{h\lambda} & \lambda^r f_t^\lambda f_h^t \end{bmatrix} = \lambda^r \begin{bmatrix} f_t^h f_\lambda^t & 0 \\ L_{h\lambda} & f_t^\lambda f_h^t \end{bmatrix}$$

$$\Rightarrow \quad (f^C)^4 = \lambda^r (f^C)^2$$

$$\Rightarrow \quad (f^C)^4 - \lambda^r (f^C)^2 = 0$$

Thus the complete lift f^C of f also has Hsu–(4,2) structure in the cotangent bundle C_{TM}.

Thus we have

Theorem 2.1. *In order that the complete lift f^C of a (1,1) tensor field f admitting Hsu-(4,2) structure in M may have the similar structure in the cotangent bundle C_{TM} it is necessary and sufficient that*

$$f_k^j f_\lambda^k L_{hj} + f_i^j f_h^i L_{j\lambda} = \lambda^r L_{h\lambda}.$$

3. Nijenhuis Tensor of Complete Lift of f^4

Nijenhuis tensor of a (1,1) tensor field f on M is given by

(3.1) $$N_{f,f}(X,Y) = [fX, fY] - f[fX,Y] - f[X,fY] + f^2[X,Y]$$

Also for the complete lift of f^4, the Nijenhuis tensor is given by

(3.2) $$\begin{aligned} N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = {} & [(f^4)^C X^C, (f^4)^C Y^C] \\ & - (f^4)^C [(f^4)^C X^C, Y^C] \\ & - (f^4)^C [X^C, (f^4)^C Y^C] \\ & + (f^4)^C (f^4)^C [X^C, Y^C], \end{aligned}$$

In view of the equation (2.1) the above equation (3.2) takes the form

$$\begin{aligned} N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = {} & [(\lambda^r f^2)^C X^C, (\lambda^r f^2)^C Y^C] \\ & - (\lambda^r f^2)^C [(\lambda^r f^2)^C X^C, Y^C] \\ & - (\lambda^r f^2)^C [X^C, (\lambda^r f^2)^C Y^C] \\ & + (\lambda^r f^2)^C (\lambda^r f^2)^C [X^C, Y^C] \end{aligned}$$

or

$$(3.3)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2)^C X^C,(f^2)^C Y^C] - (f^2)^C[(f^2)^C X^C, Y^C] - (f^2)^C[X^C,(f^2)^C Y^C] + (f^2)^C(f^2)^C[X^C,Y^C]\}.$$

We also know that ([4] page 243)

$$(3.4)\quad (f^2)^C X^C = (f^2 X)^C + \nu(\mathcal{L}_X f^2),$$

where νf has components

$$(3.5)\quad \nu f = \begin{pmatrix} 0 \\ p_a f_i^a \end{pmatrix}$$

In view of (3.4), the equation (3.3) takes the form

$$(3.6)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2X)^C,(f^2Y)^C] + [\nu(\mathcal{L}_X f^2), \nu(f^2Y)] + [(f^2X)^C, \nu(\mathcal{L}_Y f)^2] + [\nu(\mathcal{L}_X f^2), \nu(\mathcal{L}_Y f^2)] - (f^2)^C[(f^2X)^C, Y^2] - (f^2)^C[\nu(\mathcal{L}_X f^2), Y^2] - (f^2)^C[X^C,(f^2Y)^C] - (f^2)^C[X^2, \nu(\mathcal{L}_Y f^2)] + (f^2)^C(f^2)^C[X^2,Y^2]\}.$$

Let us now suppose that

$$(3.7)\quad \mathcal{L}_X f^2 - \mathcal{L}_Y f^2 = 0$$

then the equation (3.6) takes the form

$$(3.8)\quad N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{2r}\{[(f^2X)^C,(f^2Y)^C] - (f^2)^C[(f^2X)^C, Y^2] - (f^2)^C[X^C,(f^2Y)^C] + (f^2)^C(f^2)^C[X^C,Y^C]\}.$$

Let us now suppose that f acts as Hsu-structure on M[1]. Then

$$(3.9)\quad f^2 = \lambda^r I.$$

Thus the equation (3.8) becomes.

$$N_{(f^4)^C,(f^4)^C}(X^C,Y^C) = \lambda^{4r}\{[X^C,Y^C] - [X^C,Y^C] - [X^C,Y^C] + [X^C,Y^C]\} = 0.$$

Hence we have.

Theorem 3.1. *The Nijenhuis tensor of the complete lift of f^4 vanishes if the Lie derivative of the tensor field f^2 with respect to X and Y are both zero and f acts as Hsu-structure operator on M.*

4. Horizontal Lift of *HSU*-(4,2) Structure

Let f and g be two tensor fields of type (1,1) on the manifold M. If f^H denotes the horizontal lift of f, we have [5].

$$f^H g^H + g^H f^H = (fg + gf)^H \tag{4.1}$$

Taking f and g identical, we get

$$(f^H)^2 = (f^2)^H. \tag{4.2}$$

Squaring the above equation both sides and making use of the equation (4.1) we get

$$(f^H)^4 = (f^4)^H \tag{4.3}$$

Since f gives Hsu-(4,2) structure on M, we have

$$f^4 - \lambda^r f^2 = 0.$$

Taking horizontal lift in the above equation we get

$$(f^4)^H - \lambda^r (f^2)^H = 0. \tag{4.4}$$

In view of the equation (4.2) and (4.3) the above equation (4.4) takes the form

$$(f^H)^4 - \lambda^r (f^H)^2 = 0.$$

Thus the horizontal lift f^H of f also admits Hsu-(4,2) structure in the cotangent bundle C_{TM}. Hence we have

Theorem 4.1. *Let f be a tensor field of type* (1,1) *satisfying Hsu*-(4,2) *structure on the manifold M. Then the horizontal lift f^H of f also admits the same structure in the cotangent bundle C_{TM}.*

REFERENCES

[1] Duggal, K. L., *On differentiable structure defined by algebraic equation I*, Nijenhuis tensor, N. S., Vol. 22(2), pp. 238–242, 1971.

[2] Mishra, R.S., *Structure on a differentiable manifold and their applications*, Chandrama Prakashan, 50 A, Balrampur House, Allahbad, India, 1984.

[3] Yano, K., Houh, C.S. and Chen, B.Y., *Structure defined by a tensor field ϕ of type* (1,1) *satisfying* $\phi^4 \pm \phi^2 = 0$, Tensor, N. S., Vol. 23, pp. 81–87, 1972.

[4] Yano, K. and Ishihar, S., *Tangent and Cotangent bundles*, Marcel-Dekker Inc., New York, (1973).

[5] Yano, K. and Patterson, E.M., *Horizontal lift from the manifold to its cotangent bundle*. Journal Math. Soc. Japan, Vol. 19, No. 2. Pp. 185–188, 1967.

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