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On CR-Submanifolds Of A Trans Para Sasakian Manifolds

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1. Introduction

A Bejancu [1] introduced the notion of CR-submanifolds of a Kachlerian manifold. C.R–submanifolds of a Sasakian manifold have been studied by Kobayashi [4]. C.R–submanifolds of a Kenmotsu manifold have been studied by Papaghuic [2]. Oubina [5] introduced a new class of almost contact Riemannian manifold known as trans-Sasakian manifold.

The purpose of the present paper is to define and study CR-submanifolds of a trans para Sasakian manifolds.

2. Preliminaries

Let $\overline{M}$ be an n-dimensional almost para contact metric manifold with structure tensors (F, U, u, g) where F is a (1,1) tensor field, a vector field U, a 1-form u and g is an associated Riemannian metric on $\overline{M}$ which satisfy the following conditions [3]

(2.1) $$F^2 = I - u \otimes U, \quad u(U) = 1, \quad F(U) = 0, \quad uoF = 0,$$

(2.2) $$g(FX, FY) = g(X,Y) - u(X)\, u(y),$$

(2.3) $$g(FX, Y) = g(X, FY) = 0, \qquad u(X) = g(X, U), \quad \text{for all } X, Y \in \overline{M}$$

Definition : An almost para contact metric structure (F,U,u,g) on $\overline{M}$ is called trans para Sasakian if

(2.4) $$(\overline{\nabla}_X F)(Y) = \alpha\,(g(X,Y)U - u(Y)X) + \beta\,(g(FX,Y)\,U - u(Y)\,FX)$$

for α, β non zero constant and we say that trans para Sasakian structure is of type (α, β).

From the above formula, we get

(2.5) $$\overline{\nabla}_X F = \alpha FX + \beta(X - u(X)\, U).$$

Definition : A submanifold M of $\overline{M}$ is called a CR–subminifold if U is tangent to M and there exist on M a differentiable distribution D: $x \rightarrow D_x \subset T_x M$ satisfying the following conditions:

(i) D_x is invariant under F that is $FD_x \subset D_x$ for each $x \in M$,

(ii) the complimentary orthogonal distribution $D^{\perp}: x \to D_x^{\perp} \subset T_x M$ is totally real under F, that is $FD_x^{\perp}: x \subset T_x^{\perp} M$ for each $x \in M$, where $T_x M$ and $T_x^{\perp} M$ are tangent and normal space of M at x respectively.

M is an invariant (resp. anti-invariant) submanifold of $\overline{M}$ when $\dim D^{\perp}=0$ (resp. $\text{Dim}D = 0$), where D (resp. $D^{\perp}$) is the horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called U-horizontal (U-vertical) if $U_x \in D_x$ (resp. $U_x \in D_x^{\perp}$) for each $x \in M$.

For a vector field X tangent to M, we put

(2.6) $$X = PX + QX,$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. Also for a vector field N normal to M, we put

(2.7) $$FN = BN + CN,$$

where BN (resp. CN)denotes the tangential (resp. Normal) component of FN.

The Gauss and Weingarten formulas are given by

(2.8) $$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad X, Y \in TM, N \in T^{\perp}M,$$

where $\nabla^{\perp}$ is the normal connection, h (resp. A) is the second fundamental form (resp.tensor) of M in M satisfying.

(2.9) $$g(A_N X, Y) = g(h(X,Y), N).$$

If we denote the orthogonal component of $FD^{\perp}$ in $TM^{\perp}$ by μ, then we have $T^{\perp}M = FD^{\perp} \oplus \mu$, it is obvious that $F\mu = \mu$.

3. Some Basic Lemmas

Lemma 3.1. Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then we have

(3.1) $$P\nabla_X FPY - PA_{FQY}X = FP\nabla_X Y + \alpha g(X,Y)PU + \beta g(FPX,Y)PU - \alpha u(Y)PX - \beta u(Y)FPX,$$

(3.2) $$Q\nabla_X FPY - QA_{FQY}X = Bh(X,Y) + (\alpha g(X,Y) + \beta g(FQX,Y)QU - \alpha u(Y)QX.$$

(3.3) $$h(X, FPY) + \nabla_X^{\perp} FQY = FQ\nabla_X Y + Ch(X,Y) - \beta u(Y) FQX,$$

for any $X, Y \in TM$

Proof: From equations (2.4), (2.6), (2.7) and (2.8), we have

$$\overline{\nabla}_X FX - \overline{\nabla}_X Y = \alpha(g(X,Y)U + u(Y)X) + \beta(g(FX,Y)U - u(Y)FX)$$

or

$$\nabla_X FPY + h(X,FPY) + \nabla_X^{\perp} FQY - A_{FQY}X - F\nabla_X Y - Fh(X,Y)$$
$$= \alpha g(X,Y)PU + \alpha g(X,Y)QU - \alpha u(Y) PX - \alpha u(Y)QX$$
$$+ \beta g(FPX+FQX, Y)U - \beta u(U)FPX - \beta u(y)FQX.$$

or $P\nabla_X FPY + Q\nabla_X FPY + h(X, FPY) + \nabla^{\perp}_X FQY - PA_{FQY}X - QA_{FQY}X$

$$= FP\nabla_X Y + FQ\nabla_X Y + Bh(X,Y) + Ch(X,Y) + \alpha g(X,Y)PU + \alpha g(X,Y)QU$$
$$-\alpha u(Y)PX - \alpha u(Y)QX + \beta g(FPX,Y)PU + \beta g(FQX,Y)QU - \beta u(Y)FPX - \beta u(Y)FQX.$$

Now equating the horizontal, vertical and normal components we get the results

Definition: The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 3.1. Let M be a U-horizontal CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then the distribution D is parallel if and only if

(3.4) $$h(X, FY) = h(FX, Y) = Fh(X, Y), \text{ for all } X, Y \in D.$$

Proof: parallel distribution is involutive, that is

(3.5) $$h(X, FY) = h(FX, Y), \text{ for all } X, Y \in D.$$

From (3.3) and (3.5), we have

(3.6) $$h(X, FY) = Ch(X, Y).$$

Also $\nabla_X FY \in D$, $\nabla_Y FX \in D, \forall X, Y \in D$, so from (3.2) and using D-parallelness, we get

$$Bh(X,Y) = 0, \forall X, Y \in D.$$

From (2.7), we get

$$Fh(X,Y) = Bh(X,Y) + Ch(X,Y).$$

From (3.6), $Bh(X,Y) = 0$ and the above equation, we get

$$Fh(X,Y) = Ch(X,Y) = h(X, FY), \forall X, Y \in D.$$

Which proves (3.4).

Definition : A CR-submanifold M of a trans para Sasakian manifold $\overline{M}$ is said to be mixed totally geodesic if $h(X,Y) = 0$, for $X \in D$ and $Y \in D^{\perp}$

A CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$.

Definition : A normal vector field $N \neq 0$ is D-parallel normal section if $\nabla^{\perp}_x N = 0$, for all $X \in D$.

Proposition 3.2. Let M be a mixed totally geodesic U-vertical CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then the normal section $N \in FD^{\perp}$ is a D-parallel if and only if $\nabla_X FN \in D$, for all $X \in D$.

Proof: Let $N \in FD^{\perp}$ and as M be a mixed totally geodesic, we have

$$\nabla_X (FN) = \overline{\nabla}_X (FN)$$
$$\nabla_X (FN) = (\overline{\nabla}_X F)N + F\nabla_X N$$

(3.7) $$\nabla_X(FN) = F\nabla_x^{\perp} N - A_N NX.$$

Let normal section be D-parallel means $\nabla_x^{\perp} N = 0$. Let we have $A_N X \in D$ and $\nabla_x^{\perp} N = 0$ then from equation (3.7), we get $\nabla_X FN \in D$, for all $X \in D$. Conversely, we have $A_N X \in D$ and $\nabla_X FN \in D$, then from (3.7), we get $\nabla_x^{\perp} N = 0$, for all $X \in D$. This implies that normal section N is D-parallel.

This proves our assertion.

4. Integrability of Distributions of CR-submanifold:

Lemma 4.1. Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. Then we have

(4.1) $$A_{FY}Z - A_{FZ}Y + \alpha(u(Z)Y - u(Y)Z) = FP[Y,Z], \text{ for any } Y,Z \in D^{\perp}.$$

Proof: We have

$$\overline{\nabla}_{FY} FZ = (\overline{\nabla}_Y F)(Z) + F + \overline{\nabla}_Y Z.$$

Using (2.4) in the above equation, we get

$$\overline{\nabla}_Y FZ = \alpha(g(Y,Z)U - u(Z)Y) + \beta(g(FY,Z)U - u(Z)FY) + F\nabla_Y Z + Fh(Y,Z)$$
$$= \alpha(g(Y,Z)U - u(Z)Y) + FP\nabla_Y Z + Bh((Y,Z) + Ch(Y,Z) - \beta u(Z)FQY.$$

In view of (2.8) and the above equation, we have

(4.2) $$-A_{FZ}Y + \nabla_Y^{\perp} FZ = \alpha(g(Y,Z)U - u(Z)Y) - \beta u(Z)FQY + FP\nabla_Y Z + $$
$$+ FQ\nabla_Y Z + Bh(Y,Z) + Ch(Y,Z), \text{ for all } Y, Z \in D^{\perp}.$$

From (3.3), for all Y, $Z \in D^{\perp}$, we have

(4.3) $$\nabla_Y^{\perp} FZ = FQ\nabla_Y Z + Ch(Y,Z) + \beta u(Z)FQY.$$

Now from (4.2) and (4.3), we have

$$FP\nabla_Y Z = -A_{FZ}Y - \alpha g(Y,Z)U + \alpha u(Z)Y - Bh(Y,Z).$$

Similarly we have

$$FP\nabla_Z Y = -A_{FY}Z - \alpha g(Y,Z)U + \alpha u(Y)Z - Bh(Y,Z).$$

Thus from the above two equations, we get

$$FP[Y,Z] = A_{FY}Z - A_{FZ}Y + \alpha(u(Y)Z)Y - u(Y)Z, \text{ for all } Y, Z \in D^{\perp}.$$

Theorem 4.1. *Let M be a CR-submanifold of a trans para Sasakian manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable if and only if*

(4.4) $$A_{FY}Z - A_{FZ}Y = \alpha(u(Y)Z - u(Z)Y), \text{ for all } Y, Z \in D^{\perp}.$$

Proof: Suppose the distribution $D^{\perp}$ is integrable, then $[Y,Z] \in D^{\perp}$ for any $Y,Z \in D^{\perp}$. This gives $P[Y,Z] = 0$ and from (4.1) we get (4.4).

Conversely suppose (4.4) holds. Then by (4.1) we have $FP[Y, Z] = 0$ for any $Y, Z \in D^{\perp}$. From this we have $P[Y, Z] = 0$, which is equivalent to $[Y, Z] \in D^{\perp}$ for all $Y, Z \in D^{\perp}$. This implies that $D^{\perp}$ is integrable.

Theorem 4.2. *Let M be a U-horizontal CR-submanifold of a trans para Sasakian manifold* $\overline{M}$. *The distribution D is integrable if and only if*

$$h(X,FY) = h(Y,FX), \text{ for all } X, Y \in D.$$

Proof: From (3.3) for all $X, Y \in D$, we have

(4.5) $$h(X,FY) = FQ\, \nabla_X Y + Ch(X,Y).$$

Similarly, we have

(4.6) $$h(Y, FX) = FQ\, \nabla_Y X + Ch(X,Y).$$

From (4.5) and (4.6), we get

(4.7) $$h(X, FY) - h(Y, FX) = FQ\,[X,Y].$$

As the distribution D is integrable, that is, $Q\,[X,Y] = 0$.

Using this in equation (4.7), we get the result.

Conversely we have

$$h(X, FY) = h(Y, FX)$$

From equation (4.7) and above, we get

$$FQ\,(X,Y) = 0 \Rightarrow Q\,(X,Y) = 0$$

$$\Rightarrow D \text{ is integrable}$$

This proves our assertion.

Now from (2.5), we have

(4.8) $$\nabla_X U + h(X,U) = \alpha\, FPX + \alpha\, FQX + \beta\,(X - u\,(X)U).$$

From (4.8), we get the following two equations

(4.9) $$\nabla_X U = \alpha\, FPX + \beta\,(X - u\,(X)U),$$

(4.10) $$h(X,U) = \alpha\, FQX.$$

Now from (4.9) and (4.10), we get the following two relations:

(4.11) $$\nabla_X \mathrm{U} = \beta\,(X - u\,(X)U), \text{ for } X \in D^{\perp},$$

(4.12) $$h(X,U) = 0, \text{ for } X \in D.$$

Definition: A CR-submanifold of a trans para Sasakian manifold $\overline{M}$ is called D-umbilic (resp. $D^{\perp}$ umbilic) if $h(X,Y) = g(X,Y)H$ holds for all $X, Y \in D$ (resp. $X, Y \in D^{\perp}$), where H is a mean curvature vector field.

Proposition 4.1. Let M be a D-umbilic U-horizontal CR-submanifold of trans para Sasakian manifold $\overline{M}$, then M is D-totally geodesic.

Proof: Let M be a D-umbilic U-horizontal CR-submanifold, that is $h(X,Y) = g(X,Y)H$, for all $X, Y \in D$.

Putting $Y = u$, we have

$$h(X,U) = g(X,U)\, H$$

Using (4.12) in the above equation, we get

$$H = 0 \Rightarrow h(X,Y) = 0.$$

This shows that M is D-totally geodesic.

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