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On double Nörlund summability of Fourier–Jacobi series

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Abstract: In this paper we have proved a theorem on double Nörlund summability of Furier-Jacobi series, which generalizes various known results. However our theorem is as follows:

Theorem: *Let (N, p_m, q_n) be a double Nörlund method defined by a real non-negative, non-increasing sequenc $\{p_m\}$* and a real-non-negative, non-decreasing sequence $\{q_n\}$.

Let $\psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t, such that

$$\psi(n)\log n = O[\lambda(P_n)]$$

$$q_n P_n = O[(p*q)_n \log n]$$

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k} = 0\left(\frac{(p*q)_n^{1-c}}{q_n n^{(2\alpha+1)/2}}\right)$$

as $n\to\infty$, where c is a parameter with the restriction that $0 \le c \le 1$. If

$$F_1(t) = \int_0^t |F(\phi)|\, d\phi = 0\left(\frac{t^{(2\alpha+2)}\psi(1)}{\lambda(P_\tau)}\right)$$

as $t \to 0$, where $\tau = [1/t]$ then the Fourier-Jabobi series is summable (N, p_m, q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2,\ \beta > -1/2$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)|\, dx < \infty$$

are satisfied, where b is fixed.

Definitions and Notations: Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$ such that the integral.

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\,dx \tag{1.1}$$

exists in the sense of Lebesgue for $\alpha > -1$, $\beta > -1$. The Fourier-Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{1.2}$$

where

$$a_n = g_n \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) f(x)\,dx \tag{1.3}$$

with

$$g_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined by the generating function.

$$2^{\alpha+\beta}(1-2xt+t^2)^{-1/2}[1-t+(1-2xt+t^2)^{1/2}]^{-\alpha}\,[1+t+(1-2xt+t^2)^{1/2}]^{-\beta}$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n \tag{1.5}$$

Let us write

$$F(\phi) = \{f(\cos\phi) - A\}(\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1}$$

A being a fixed constant.

Let $\{s_n\}$ be the sequence of partial sums of a given infinite series $\sum a_n$. Let $\{p_m\}$ and $\{q_n\}$ be any two sequences of constants with P_m and Q_n as their partial sums respectively and let

$$(p * q)_n = \sum_{k=0}^{n} p_{n-k}\,q_k = \sum_{k=0}^{n} p_k\,q_{n-k} \tag{1.6}$$

tends to infinity as $n \to \infty$

If the sequence-to sequence transformation defined by

$$t_n^{p,q} = \frac{1}{(p*q)^n} \sum_{k=0}^{n} p_{n-k}\,q_k\,s_k \tag{1.7}$$

tends to a fixed limit s as $n \to \infty$, then the sequence $\{s_n\}$ or the series $\sum a_n$ is said to be summable by double Nörlund method (N, p_m, q_n) to s, Borwein [2].

2. Introduction: The study of summability of Fourier-Jacobi series by ordinary Nörlund summability method has been made by several workers ([1],[5],[9],[16]). In the present paper we study the summability of Fourier-Jacobi series by double Nörlund summability method.

Dealing with the Nörlund summability of Fourier-Jacobi series, Prasad and Saxena [10] have established the follwing :

Theorem A : If

$$F_1(t) = \int_0^t |F(\phi)|\, d\phi = 0\left(\frac{\psi(t)\, t^{2\alpha+2}}{0(P_n)}\right) \quad \text{as } t \to 0$$

where

$$F(\phi) = \{f(\cos\phi) - A\}(\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1},$$

$\psi(t)$ and $\theta(t)$ are non-negative monotonic increasing functions of t such that

$$\psi(n)\log n = O(\theta(P_n)) \quad \text{as } n \to \infty \tag{2.2}$$

$$n^{(2\alpha+1)/2} = o(P_n) \quad \text{as } n \to \infty \tag{2.3}$$

and

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k} = 0\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{ as } n \to \infty \tag{2.4}$$

then the Fourier-Jacobi series (1.2) is summable (N, p_n) at the point $x = +1$, to sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2 \text{ and the antipole condition}$$

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4} |f(x)|\, dx < \infty \tag{2.5}$$

are satisfied, where b is fixed and (N, p_n) is regular Nörlund method defined by the real non-negative and non-increasing sequence $\{p_n\}$ such that $p_n \to \infty$ as $n \to \infty$.

The object of this paper is to generalize the above theorem to a more general class of double Nörlund summability of Fourier-Jacobi series.

2. We establish our result in the form of the following theorem.

Theorem: *Let (N, p_m, q_n) be a double* Nörlund *method defined by a real non-negative, non-increasing sequence $\{p_m\}$ and a real non-negative, non decreasing sequenc $\{q_n\}$.*

Let $\psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t such that

$$\psi(n)\log n = O[\lambda(P_n)] \tag{3.1}$$

$$q_n P_n = O[(p * q)_n \log n] \tag{3.2}$$

$$(3.3)\qquad \sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2}\log k} = O\left(\frac{(p*q)_n^{1-c}}{q_n\, n^{(2\alpha+1)/2}}\right)$$

as $n\to\infty$, where c is a parameter with the restriction that $0 \le c \le 1$.

$$(3.4)\qquad \text{If } F_1(t) = \int_0^t |F(\phi)|\, d\phi = 0\left(\frac{t^{(2\alpha+2)}\psi(t)}{\lambda(P_\tau)}\right)$$

as $t \to 0$, where $\tau = [\,1/t\,]$ then the Fourier-Jacobi series (1..2) is summable (N, p_m, q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2$$

and the antipole condition

$$(3.5)\qquad \int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2}\, |f(x)|\, dx < \infty$$

are satisfied, where b is fixed.

4. The following lemmas are needed for the proof of our theorem

Lemma 1 : ([15] p. 167 & 196) : For $\alpha > -1$, $\beta > -1$

$$P_n^{(\alpha,\beta)}(\cos\phi) = \begin{cases} O(n^{\alpha}), \text{ when } 0 \le \phi \le 1/n \\ O(n^{\beta}), \text{ when } \pi - 1/n \le \phi \le \pi \\ \dfrac{1}{(n\pi)^{1/2}}(\sin\phi/2)^{-(2\alpha+1)/2}(\cos\phi/2)^{-(2\beta+1)/2} \\ \quad \times\left(\cos\left\{\dfrac{(2n+\alpha+\beta+1)}{2}\phi - (2\alpha+1)\dfrac{\pi}{4}\right\} + \dfrac{O(1)}{n\sin\phi}\right) \\ \quad \text{when } 1/n \le \phi \le \pi - 1/n \end{cases}$$

Lemma 2 : The antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2}\, |f(x)|\, dx < \infty$$

is equivalent to

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2}\, |f(x) - A|\, dx < \infty$$

which is further

$$\int_{n}^{\pi} |F(\phi)|\, (\cos\phi/2)^{-\alpha-\beta-1}\, d\phi < \infty, \; 0 < n < \pi\,.$$

Lemma 3. [7]. If $\{p_n\}$ is a non-negative, non-increasing sequence then for large n, uniformly in $0 < \phi \le \pi$, $0 \le a \le b \le n$,

$$\left| \sum_{k=a}^{n} p_k \cos\{(n-k+p)\phi - \gamma\}(n-k)^{(2\alpha+1)/2} \right| = O\left(n^{(2\alpha+1)/2} \rho(1/\phi)\right)$$

where

$$\rho = \frac{\alpha+\beta+2}{2}, \quad \gamma = \frac{(2\alpha+3)\pi}{4}, \quad \alpha \ge -1/2$$

Lemma 4. [7]: If $\{p_n\}$ is a non-negative, non-increasing and $\{q_n\}$ is a non-negative, non-decreasing sequence then

$$\sum_{k=0}^{n-1} p_k q_{n-k} (n-k)^{(2\alpha-1)/2} = O\left((p * q)_n n^{(2\alpha+1)/2}\right)$$

Lemma 5: Let

$$N_n(\phi) = \frac{2^{\alpha+\beta+1}}{(p*q)_n} \sum_{k=0}^{n-1} p_k q_{n-k} \delta_{n-k} P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)$$

where

$$\delta_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1)\Gamma(n+\beta+1)} \simeq \frac{2^{-\alpha-\beta-1}}{\Gamma(n+1)}, \; n^{\alpha+1}$$

Then for $-1/2 \le \alpha < 1/2$, $\beta > -1/2$ and $\{p_m\}, \{q_n\}$ satisfying the conditions of the theorem, we have

$$N_n(\phi) = \begin{cases} O(n^{2\alpha+2}), \text{ when } 0 \le \phi \le 1/n & (4.1) \\ O(n^{2\alpha+\beta+1}), \text{ when } \pi - 1/n \le \phi \le \pi & (4.2) \\ O\left(\dfrac{q_n n^{(2\alpha+1)/2}}{(p*q)_n} (\sin\phi/2)^{-(2\alpha+3)/2} (\cos\phi/2)^{-(2\beta+1)/2}\right) & (4.3) \\ \quad + O\left[n^{(2\alpha-1)/2} (\sin\phi/2)^{-(2\alpha+5)/2} (\cos\phi/2)^{-(2\beta+3)/2}\right] \\ \quad \text{when } 1/n \le \phi \le \pi - 1/n \end{cases}$$

Proof: Using Lemma 1 for $0 \le \phi \le 1/2$ together with Lemma 4, the required estimate in (4.1) follows. For the estimate in (4.2), we use Lemma 1 for $\pi - 1/n \le \phi < \pi$ together with Lemma 4.

For $1/n \le \phi \le \pi - 1/n$, we have from Lemma 1.

$$N_n(\phi) = \frac{O(1)}{(p*q)_n} \sum_{k=0}^{(n-1)} p_k\, q_{(n-k)}\, (n-k)^{(2\alpha+1)/2}$$

$$\times (\sin\phi/2)^{-(2\alpha+1)/2}\, (\cos\phi/2)^{-(2\beta+1)/2}$$

$$\times \left(\cos\{(n-k+p)\,\phi - \gamma\} + \frac{O(1)}{(n-k)\sin\phi} \right)$$

Since for fixed $n, \{q_{n-k}\}$ is non-increasing, we can deal with the first term of the right by first using the second mean value theorem and then applying Lemma 3 to deal with the second term on the right, we apply the result of Lemma 4 and the required estimate follows.

Lemma 6. The condition

$$q_n\, n^{(2\alpha+1)/2} = O(p*q)_n^{1-c}$$

For $0 \le c \le 1$ under the hypothesis of the theorem.

Proof: The expression on the left of (3.3) is increasing and hence greater than or equal to a positive constant. Hence (3.4) implies that for some positive constant A.

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} \ge \sum_{k=a}^{n} \frac{P_k}{k^{(2\alpha+1)} \log k}$$

where $a > 2$

(note that $k\,p_k \le p_k \Rightarrow p_k \ge k\,p_k$, by the condition on $\{p_n\}$)

$$= \sum_{k=a}^{n} \frac{P_k\, q_{n-k}}{q_{n-k}\, k^{(2\alpha+3)/2} \log k}$$

$$> \frac{(p*q)_n}{q_n\, n^{(2\alpha+1)/2}} \text{ by the condition on } \{q_n\}$$

$$> \frac{(p*q)_n^{1-c}}{q_n\, n^{(2\alpha+1)/2}} \qquad \{\because (p*q)_n > (p*q)_n^{1-c} \text{ for } 0 \le c \le 1\}$$

$$> A$$

From where the result in Lemma 6 follows.

5. **Proof of the theorem:** Following Obrechkoff ([8, p. 99. and Rao [11] the n^{th} partial sum of the series (1.2) at the point $x = 1$ is given by

$$S_n(1) = 2^{\alpha+\beta+1}\, \delta_n \int_0^{\pi} (\sin\phi/2)^{2\alpha-1}\, (\cos\phi/2)^{2\beta+1}\, .\, f(\cos\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

Consequently,

$$S_n(1) - A = 2^{\alpha+\beta+1}\,\delta_n \int_0^{\pi} F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \tag{5.1}$$

Using (1.7), the (N, P_m, q_n) mean of the series (1.2) is given by

$$t_n^{p,q} - A = \frac{1}{(p * q)_n} \sum_{k=0}^{n} p_k\, q_{n-k}\, \{S_{n-k}(1) - A\}$$

$$= \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi$$

$$= I, \text{ say}$$

$$= \left(\int_0^{1/n} + \int_{1/\eta}^{\eta} + \int_{\eta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \right) F(\phi)\, N_n(\phi)\, d\phi$$

$$t_n^{p,q} - A = I_1 + I_2 + I_3 + I_4 \tag{5.2}$$

say, where η is a suitable constant such that $0 < \eta < \pi$.

Now in order to prove our theorem, we have to show that

$$I = 0\,(1), \quad \text{as } n \to \infty \tag{5.3}$$

For which we need to prove that

$$I_j = 0\,(1), \quad \text{as } n \to \infty \tag{5.4}$$

for $j = 1, 2, 3, 4, \ldots$

Let us first consider I_1, we have

$$I_1 = O\left(\int_0^{1/n} |F(\phi)|\, |N_n(\phi)|\, d\phi \right)$$

$$= O\,(n^{2\alpha+2}) \int_0^{1/n} |F(\phi)|\, d\phi \quad \text{by (4.1) of Lemma 5.}$$

$$= O\,(n^{2\alpha+2}).0\left(\frac{\psi(n)\, n^{-2\alpha-2}}{\lambda(P_n)} \right) \text{ by (3.4)}$$

$$= 0\left(\frac{1}{\log n} \right) \text{ by (3.1)}$$

$$I_1 = o(1), \quad \text{as } n \to \infty \tag{5.5}$$

Considering I_2, we have

$$I_2 = 0\left(\frac{q_n n^{(2\alpha+1)/2}}{(P*q)_n}\right)\int_{1/n}^{\eta}\frac{|F(\phi)|\,P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\,d\phi$$

$$+ 0\,(n^{(2\alpha-1)/2})\int_{1/n}^{\eta}\frac{|F(\phi)|}{\phi^{(2\alpha+5)/2}}\,d\phi$$

using (4.3) of Lemma 5.

(5.6) $$I_2 = I_{2.1} + I_{2.2}, \text{ say}$$

Given $\varepsilon > 0$, let η be chosen so that

$$|F_1(\phi)| \le \frac{\varepsilon\,\phi^{(2\alpha+2)}\,\delta_{(1/\phi)}}{\phi(P_{(1/\phi)})}, \quad 0 \le \phi \le \eta$$

Then

$$|I_{2.1}| \le \frac{Kq_n n^{(2\alpha+1)/2}}{(P*q)_n}\int_{1/n}^{\eta}\frac{|F(\phi)|\,P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\,d\phi$$

$$\le \frac{Kq_n n^{(2\alpha+1)/2}}{(P*q)_n}\left\{\left(\frac{|F_1(\phi)|\,P_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}\right)_{1/n}^{\eta} - \int_{1/n}^{\eta}F_1(\phi)\,d\left(\frac{P_{(1/\phi)}}{\phi^{(2\alpha+3)/2}}\right)\right\}$$

(5.7) $$|I_{2.1}| = I_{2.1.1} + I_{2.1.2} \text{ (say)}$$

whre K is an absolute constant, not necessarily same at each occurrence. If $K(\eta)$ denotes a constant depending on η, we see that, for fixed η,

$$|I_{2.1.1}| = K(\eta)\frac{q_n n^{(2\alpha+1)/2}}{(p*q)_n} + O\left(\frac{q_n p_n \psi(n)}{(p*q)_n \lambda(P_n)}\right)$$

$$= K(\eta)\,o\left(\frac{1}{(p*q)_n^c}\right) + o(1)$$

(5.8) $$|I_{2.1.1}| = o(1), \text{ as } n \to \infty$$

by using Lemma 6, (3.4), (3.1) and (3.2).

Further

$$I_{2.12} \leq \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \int_{1/n}^{\eta} \frac{\phi^{2\alpha+2}\,\psi_{(1/\phi)}}{\lambda\,(P_{[1/\phi]})} \left| d\left(\frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}}\right)\right|$$

$$\leq \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \int_{1/\eta}^{n} \frac{x^{-2\alpha-2}}{\log x}\, d\,|\,P_{[x]}\, x^{(2\alpha+3)/2}\,|$$

Using (3.1)

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \left(\int_{1/\eta}^{n} \frac{x^{-(2\alpha+1)/2}}{\log x}\, d\,P_{[x]} + \frac{(2\alpha+3)}{2} \int_{1/\eta}^{n} \frac{x^{(-2\alpha+3)/2}}{\log x}\, P_{[x]}\, dx \right)$$

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p*q)_n} \left(L + \frac{(2\alpha+3)}{2} M \right), \text{ say}$$

Now,

$$L = \sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k}$$

$$L = \mathrm{O}\left(\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+3)/2} \log k} \right)$$

and

$$M \leq \sum_{k=1}^{n-1} P_k \int_{k}^{k+1} \frac{x^{-(2\alpha+3)/2}}{\log x}\, dx$$

$$= \mathrm{O}\left(\sum_{k=1}^{n-1} \frac{P_k}{k^{(2\alpha+3)/2} \log k} \right)$$

Hence

$$|\,I_{2.12}\,| \leq \frac{K\,\varepsilon}{(p*q)_n^{c}} = \mathrm{o}(1), \text{ as } n \to \infty \tag{5.9}$$

by using (3.3) from (5.7), (5.8) and (5.9) it follows that

$$I_{2.1} = \mathrm{o}\,(1), \text{ as } n \to \infty \tag{5.10}$$

Next, considering $I_{2.2}$, we have

$$|I_{2.2}| \leq K n^{(2\alpha-1)/2} \int_{1/n}^{\eta} |F(\phi)|\, \phi^{-(2\alpha+5)/2}\, d\phi$$

$$= n^{(2\alpha-1)/2} \left\{ K\,[F_1(\phi)\, \phi^{-(2\alpha+5)/2}]_{1/n}^{\eta} + K \int_{1/n}^{\eta} F_1(\phi)\, \phi^{-(2\alpha+7)/2}\, d\phi \right\}$$

$$|I_{2.2}| = I_{2.2.1} + I_{2.2.2}, \text{ say} \tag{5.11}$$

Hence

$$|I_{2.2.1}| = K(\eta)\, n^{(2\alpha+1)/2} + o(1)$$

$$|I_{2.2.1}| = o(1), \text{ as } n \to \infty \tag{5.12}$$

and

$$|I_{2.2.2}| \leq K \varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{\phi^{(2\alpha+2)}\, \psi_{(1/\phi)}\, \phi^{-(2\alpha+7)/2}}{\lambda(P_{[1/\phi]})}\, d\phi$$

$$= K \varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{x^{-(2\alpha+1)/2}}{\log x}\, dx$$

$$|I_{2.2.2}| \leq K \varepsilon \text{ ; since } \alpha < 1/2. \tag{5.13}$$

Hence from (5.11), (5.12) and (5.13), it follows that

$$I_{2.2} = o(1), \text{ as } n \to \infty \tag{5.14}$$

Thus from (5.6), (5.10) and (5.14), we have

$$I_2 = o(1), \text{ as } n \to \infty \tag{5.15}$$

Considering I_3, we have

$$I_3 = O\left(\frac{(p*q)_n n^{(2\alpha+1)/2}}{(p*q)_n} \right) \int_{\eta}^{\pi-1/n} |F(\phi)|\, (\sin\phi/2)^{-(2\alpha+3)/2} (\cos\phi/2)^{-(2\beta+1)/2}\, P_{[1/\phi]}\, d\phi$$

$$+ O(n^{(2\alpha+1)/2}) \int_{\eta}^{\pi-1/n} |F(\phi)|\, (\sin\phi/2)^{-(2\alpha+5)/2}\, (\cos\phi/2)^{-(2\alpha+3)/2}\, d\phi$$

$$I_3 = I_{3.1} + I_{3.2}, \text{ say} \tag{5.16}$$

Since $(\sin\phi/2)^{-(2\alpha+3)/2}$ is bounded for $\eta \leq \phi \leq \pi$ and since $P_{[1/\phi]}$ is bounded and $-\beta - 1/2 > -\beta - \alpha - 1$, , we have

$$I_{3.1} = 0\left(\frac{q_n n^{(2\alpha+1)/2}}{(p*q)_n}\right)\int_{\eta}^{\pi-1/n} |F(\phi)|\,(\cos\phi/2)^{-\alpha-\beta-1}\,d\phi$$

$$= 0\left(\frac{q_n n^{(2\alpha+1)/2}}{(p*q)_n}\right) \quad \text{by lemma 2.}$$

$$= 0\left(\frac{1}{(p*q)_n^{c}}\right) \quad \text{by lemma 6.}$$

(5.17) $$I_{3.1} = o(1), \quad \text{as } n \to \infty.$$

Again

(5.18) $$I_{3.2} = O(n^{(2\alpha-1)/2})\left(\int_{\eta}^{\eta'} + \int_{\eta'}^{\pi-1/n}\right), \quad \text{say}$$

Given $\varepsilon' > 0$, we can choose η' so that

$$\int_{\eta'}^{\pi} (\cos\phi/2)^{-\alpha-\beta-1}\,|F(\phi)|\,d\phi \le \varepsilon'.$$

The contribution of $I_{3.2}$ of the range $(\eta', \pi - 1/n)$ is

$$\le K n^{(2\alpha-1)/2} \int_{\eta'}^{\pi-\frac{1}{n}} |F(\phi)|\,(\cos\phi/2)^{-(2\beta+3)/2}\,d\phi$$

$$\le K n^{(2\alpha-1)/2} \int_{\eta'}^{\pi-1/n} |F(\phi)|\,(\cos\phi/2)^{-\alpha-\beta-1}\,(\cos\phi/2)^{-(2\alpha-1)/2}\,d\phi$$

(5.19) $$\le K\varepsilon';$$

Since, in the range considered,

$$(\cos\phi/2)^{-(2\alpha-1)} = O\left(\frac{1}{n^{(2\alpha-1)/2}}\right)$$

Thus the lim sup of the contribution of this range can be made arbitrarily small by suitable choice of ε'. Thus it is enough to prove that for fixed η', the contribution in the range (η, η') is zero. For fixed η',

$$\int_{\eta}^{\eta'} |F(\phi)|\,(\sin\phi/2)^{-(2\alpha+5)/2}\,(\cos\phi/2)^{-(2\beta+3)/2}\,d\phi$$

is a constant, so that the contribution

$$= O[n^{(2\alpha-1)/2}]$$

(5.20) $$= o(1), \text{ as } n \to \infty \text{ for } \alpha < 1/2$$

From (5.18), (5.19) and (5.20), it is obtained that

(5.21) $$I_{3,2} = o(1), \text{ as } n \to \infty.$$

Hence from (5.16), (5.17) and (5.21), it is obtained that

(5.22) $$I_3 = o(1), \text{ as } n \to \infty.$$

Finally, considering I_4, we see that

$$I_4 = o(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |F(\phi)|\, dx,$$

by (4.2) of Lemma 5.

But $$n^{\alpha+\beta+1} = o[(\cos\phi/2)^{-\alpha-\beta-1}]$$

uniformly in $\pi - 1/n \le \phi \le \pi$; whence by the use of Lemma 2, it follows immediately that

(5.23) $$I_4 = o(1), \text{ as } n \to \infty.$$

Collecting (5.5), (5.15), (5.22) and (5.23) the required result in (5.4) is established, which, in turn, proves the result in (5.3).

This completes the proof of the theorem.

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