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On Hypersurfaces of H Hsu - Manifold

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Abstract: In these papers [2], [3] and [4] we have studied some properties of hypersurfaces of *H* Hsu - manifold. In this paper we have defined hyperbolic almost kahler manifold and studied its hypersurfaces. It has been found that the hypersurface of hyperbolic almost Kahler manifold is locally quasi-Sasakian manifold. Some results regarding the hypersurfaces of a flat *H* Hsu-manifold have also been obtained.

Keywords: Hyperbolic Almost Kahler manifold, Curvature tensor, Reimannian connection.

1. Introduction

We consider a differentiable manifold M^n of class C^∞. Let there be a vector valued linear function F of C^∞, satisfying the algebraic equation

(1.1) $$F^2 = -a^r I_n$$

where 'a' is a complex number.

Then F is said to give to M^n a hyperbolic differentiable structure, briefly H Hsu-structure, defined by algebraic equation (1.1) and the manifold M^n is called HHsu-manifold [5]. The equation (1.1) gives different algebraic structures for different values of a. If $a \neq 0$, it is a hyperbolic π-structure, $a = \pm 1$, it is an almost complex or an almost hy6perbolic product structure. $a = \pm 1$, it is an almost product or an almost hyperbolic complex structure and $a = 0$, it is an almost tangent or a hyperbolic almost tangent structure. In the second case n has to be even and in the second and third cases $a^{2r} = 1$.

If the H Hsu-structure is endowed with Hermite metric G. such that

(1.2) $$G(F\lambda, F\mu) = a^r G(\lambda, \mu)$$

Then $\{F,G\}$ is said to give to M^n hyperbolic Hermitee structure, briefly known as H Hsu-structure subordinate to H Hsu-structure.

In a hyperbolic H-structure, if

(1.3) $$(E_\lambda, F)(\mu) = 0 \text{ or } (E_\lambda, F)(F\mu) = 0$$

is satisfied, then M^n is said to be a hyperbolic Kahler manifold. E is the Reimannian connexion.

If a hyperbolic H-structure, if

(1.4) $$(E_\lambda, F)(\mu) + (E_\mu F)(\lambda) = 0$$

is satisfied, then M^n is said to be a hyperbolic nearly Kahler manifold.

Let us consider M^n and M^n as the H Hsu-manifold and its hypersurface respectively. Let $b : M^m \to M^n$ be the embedding map, such that $p \in M^m \Rightarrow bp \in M^n$,

Let B be the corresponding Jacobian map such that a vector field X in M^n at p, BX in M^n at bp. Let g be the induced Reimannian metric in M^m. Thus we have

(1.5) $$G(BX, BY)ob = g(X, Y)$$

for arbitrary vector fields X, Y in M^n.

(1.6a) $$G(N, N)ob = 1$$

(1.6b) $$G(N, BX)ob = 0$$

for a unit normal to M^m.

If we put

(1.7a) $$FBX = B(fX) + u(X)N$$

(1.7b) $$FN = -BU$$

Then it can be easily seen that

(1.8a) $$\overline{\overline{X}} = a^r X + u(X)U$$

(1.8b) $$u(fX) = 0$$

(1.8c) $$u(U) = a^r$$

(1.8d) $$fU = 0 \text{ and}$$

(1.9) $$g(\overline{X}, \overline{Y}) = a^r g(X, Y) - u(X)u(Y)$$

where $X \underset{=}{def} fX$ and $u(X) = g(X, U)$

i.e. the induced structure in a general contact metric structure.

Let E and D be the Reimannian connexions in M^n and M^m respectively, Gauss and Weingarten equations are

(1.10a) $$E_{BX} BY = BD_X Y + {}'H(X, Y)N$$

(1.10b) $$E_{BX} N = -BHX, \quad \text{respectively [1].}$$

where

$$H(X, Y) \underset{=}{def} g(HX, Y)$$

Let R and K denote the curvature tensors with respect to the connexions E and D respectively. The generalized Gauss and Mainardi-Codazzi equations are given by [5].

(1.11a) $$'R(BX,BY,BZ,BW)\,o\,b = {}'K(X,Y,Z,W) + {} + a^r\,'H(X,Z)'H(Y,W) - a^r\,'H(Y,Z)'H(X,W)$$

(1.11b) $$'R(BX,BY,BZ,N)\,o\,b = a^r\{(D_X H)(Y,Z) - (D_Y H)(X,Z)\}$$

where

$$'R(BX,BY,BZ,BW) \underset{=\!=}{def} G(R(BX,BY,BY),BW)$$

On the hypersurface of a hyperbolic Kahler manifold subordinate of H Hsu-manifold the following results hold [2].

(1.12a) $$(D_X f)Y = u(Y)HX - 'H(X,Y)U$$

(1.12b) $$(D_X u)(Y) = -'H(X,\bar{Y})$$

Agreement (1.1): In the above and sequal $\lambda, \mu, \nu \ldots$ will be taken as arbitrary vector fields in the enveloping manifold and $X, Y, Z, \ldots$ as arbitrary vector fields in the hypersurface.

2.Hyperbolic Almost Kahler Manifold

Definition (2.1): Hyperbolic Hermite manifold satisfying

(2.1a) $$(E_\lambda F)(\mu,\nu) + (E_\mu F)(\nu,\lambda) + (E_\nu F)(\lambda,\mu) = 0$$

where $$'F(\lambda,\mu) \underset{=\!=}{def} G(F\lambda,\mu)$$

Will be called hyperbolic almost Kahler manifold, sub ordinate to H Hsu-manifold.

From the equation (1.7a), we have

(2.1)b) $$G(FBX, BY) = G(BfX,BY) + u(X)\,G(N, BY)$$

Differentiating equation (2.1b), covariantly with respect to BZ, then using the equations (1.5), (1.6), (1.7a) and (1.10)a), we have

(2.2) $$(E_{BZ}'F)(BX,BY)ob = (D_Z,'f)(X,Y) + 'H(X,Z)u(Y) - 'H(Y,Z)u(X)$$

Writing two other equations by cyclic permutations of X, Y, Z, we have

(2.3) $$(E_{BY}'F)(BZ,BX)ob = (D_Y,'f)(Z,X) + 'H(Z,Y)u(X) - 'H(X,Y)u(Z)$$

and

(2.4) $$(E_{BX}'F)(BY,BZ)ob = (D_X,'f)(Y,Z) + 'H(Y,X)u(Z) - 'H(X,Z)u(Y)$$

Thus we have the following theorem:

Theorem (2.1): *If the enveloping manifold is a hyperbolic almost Kahler manifold, its hypersurface is given by*

(2.5) $(D_X,'f)(Y,Z)+(D_Y,'f)(Z,X)+(DZ,'f)(X,Y)=0$

Proof: Adding the equations (2.2), (2.3) and (2.4), we get

(2.6) $$\{(E_{BZ}'F)(BX,BY)+(E_{BY}'F)(BZ,BX)+(E_{BX}'F)(BY,BZ)\}ob$$
$$=(D_Z,'f)(Y,Z)+(D_Y,'f)(Z,X)+(D_Z,'f)(X,Y)$$

Using the equation (2.1)a) in the equation (2.6), we get the equation (2.5).

Corollary (2.1): Hypersurface of Hyperbolic almost Kahler manifold is locally Quassi Sasakian manifold.

Proof: Equation (2.5) proves the statement.

Theorem (2.2): *For the hypersurface of hyperbolic almost Kahler manifold, we have*

(2.7) $$(D_X,'f)(\overline{Y},\overline{Z})+(D_Y,'f)(\overline{Z},\overline{X})+(D_Z,'f)(\overline{X},\overline{Y})+$$
$$+'f((D_Z,f)X-(D_X,f)Z,\overline{Y})+'f((D_Y,f)Z-(D_Z,f)Y,\overline{X})+$$
$$+'f((D_X,f)Y-(D_Y,f)X,\overline{Z})=0$$

Proof: We have

(2.8a) $$'f(X,Y)=g(\overline{X},Y)=-'f(X,Y)$$

and

(2.8b) $$'f(\overline{X},\overline{Y})=a^r\,'f(X,Y)$$

Differentiating (2.8)b) covariantly with respect to Z and Using the equation (2.8) again,
We get

(2.9a) $$(D_Z'f(\overline{X},\overline{Y})+'f((D_Zf)(X,\overline{Y})+'f(\overline{X},(D_Zf)Y)=a^r(D_Z'f)(X,Y)$$

Similarly, writing two other equations, we have

(2.9b) $$(D_Y\,'f(\overline{Z},\overline{X})+'f((D_Yf)Z,\overline{X})+'f(\overline{Z},(D_Yf)X)=a^r(D_Y\,'f)(Z,X)$$

(2.9c) $$(D_X\,'f)(\overline{Y},\overline{Z})+'f((D_Xf)Y,\overline{Z})+'f(\overline{Y},(D_Xf)Z)=a^r(D_Z\,'f)(X,Y)$$

Adding the equations (2.9a), (b) and (c) then using the equations (2.8a) and (2.5), we get the required result.

3. Hypersurfaces of Flat *H* Hsu-manifold

Theorem (3.1): *The umbilical hypersurface of a hyperbolic General Differentiable (H Hsu) manifold is of constant Reimannian curvature, iff the enveloping manifold is* flat.

Proof: Let the hypersurface be umbilic, i.e.

$$'H(X,Y)=g(X,Y) \qquad [1],$$

then (1.11)a), gives

$$'R(BX, BY, BZ, BW)\, ob = 'K(X,Y,Z,W) + + d^{r} g(X,Y)\, g(Y,W) - d^{r} g(Y,Z)\, g(X,W) \tag{3.1}$$

If the enveloping manifold is flat, then (3.1) reduces to

$$'K(X,Y,Z,W) = d^{r}\{g(Y,Z)\, g(X,W) - g(X,Z)\, g(Y,W)\} \tag{3.2}$$

This shows that the hypersurface is constant Reimannian Curvature.

Conversely, if (3.2) holds, then using (3.2) in (3.1), we have $'R(BX, BY, BZ, BW) = 0$, that is the manifold is flat.

Theorem (3.2): *The scalar curvature of the umbilical hypersurface M^n of a flat H Hsu-manifold M^n is given by*

$$r = m\,(m-1)\, d^{r} \tag{3.3}$$

Proof: The unbilical hypersurface is of constant R reimannian curvature (by theorem (3.1)). We have

$$K(X,Y,Z,) = d^{r}\{g(Y,Z)\, X - g(X,Z) Y\}$$

From this we at once get the equation (3.3).

Theorem (3.3): *The quasi-umbilical hypersurface of a flat H Hsu-manifold can never be of constant Reimannian cuevature.*

Proof: Let the hypersurface of a flat H Hsu-manifold be quasi-umbilical, then we can always write.

$$'H(X,Y) = g(X,Y) + u(X)u(Y) \tag{3.4}$$

Using (3.4) in (1.11a), we have

$$'R(BX, BY, BZ, BW)\, ob = 'K(X,Y,Z,W) + + d^{r} g(X,Z)\, g(Y,W) - d^{r} g(Y,Z)\, g(X,W) \tag{3.5}$$

$$d^{r} g(Y,W) u(X)\, u(Z) + d^{r} g(X,Z)\, u(Y)\, u(W) - d^{r} g(Y,Z)\, u(X)\, u(W)$$

Now,

$$'K(X,Y,Z,W) = d^{r}\{g(Y,Z)\, g(X,W) - g(X,Z)\, g(Y,W)\}$$

If $\quad d^{r}\{g(Y,W)\, u(X)\, u(Y)\, u(Z) + g(X,Z)\, u(Y)\, u(W) - g(Y,Z) u(X)\, u(W) - g(X,W) u(Y)\, u(Z)\} = 0$

Let $d^{r} \neq 0$ then

$$\{g(Y,W) u(X)\, u(Z) + g(X,Z) u(Y) u(W) - g(Y,Z) u(X)\, u(W) - g(X,W) u(Y)\, u(Z)\} = 0$$

or $\quad g(Y,W)\, u(X)\, U + u(Y)\, u(W)\, X - u(X)\, u(W) Y - g(X,W)\, u(Y)\, U = 0$

or $\quad d^{r} g(Y,W) + mu(Y)\, u(W) - u(Y)\, u(W) - u(Y)\, u(W) = 0$

or $\quad d^{r} g(Y,W) + (m-2)\, u(Y)\, u(W) = 0$

or $\quad d^{r}\, Y + (m-2)\, u(Y)\, U = 0$

or $$ma^r + (m-2)a^r = 0$$

or $$2a^r(m-1) = 0$$

or $$a^r = 0$$

But $a^r \neq 0$, thus the quasi-umbilical hypersurface can not be of constant Reimannian curvature.

Theorem (3.4): *If the hypersurface of a flat H Hsu-manifold of minimal variety, then* $div\, H = 0$.

But the converse is not true in general.

Proof: Let the hypersurface be of minimal variety, then

$$\text{tr. } H = 0\,, [1]$$

Since the enveloping manifold is flat, equation (1.11b) implies that

$$(D_X H)Y - (D_Y H)X = 0$$

Contracting this equation, we get

$$(\text{div } H)Y = Y \text{ tr. } H = 0 \quad (\text{since Tr. } H = 0)$$

Conversely, if div. $H = 0$, then from the last equation, we get

$$Y \text{ tr. } H = 0 \qquad \text{i.e.} \qquad \text{tr. } H = \text{constant}$$

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