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On Induced Structures and Curvature Tensors In the Tangent Bundle

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Abstract: In this paper, we have studied structures induced In the tangent bundle T(M) if the base space M admits almost GF-contact structure. Certain results on curvature tensors on T(M) are also studied assuming that M admits the Riemannian connection.

1. Preliminaries :

Let M be an n-dimensional differentiable manifold and *T(M)*denotes tangent bundle of M. Then *T(M)* is also a differentiable nanifold of dimension $2n$ [3]

Suppose over the basespace M, there exists a tensorfield F of type (1,1), a vectorifield ξ and a 1-form η satisfying.

$$\text{(1.1)} \qquad \text{(i) } F^2 = a^2 I_n + \eta \otimes \xi \qquad \text{(ii) } F\xi = 0 \quad \text{and} \qquad \text{(iii) } \eta(\xi) = -a^2$$

'a' any complex number not zero. Then we say that the basepace M admits almost GF-contact structure.

(1.2) If $$F^2 = a^2 I_n ,$$

we say that *M* admits *GF*- Structure [2].

It is well known that F^C, F^V, F^H etc are complete, vertical and horizontal lifts in *T(M)* of (1.1) tensorfield F or *M* if we define

$$\text{(1.3)} \qquad P = F^C + \frac{1}{a}\eta^V \otimes \xi^V + \frac{1}{a}\eta^C \otimes \xi^C$$

and

$$\text{(1.4)} \qquad Q = F^C + \frac{1}{a}\eta^V \otimes \xi^V + \frac{1}{a}\eta^H \otimes \xi^H$$

then it is easy to show that P and Q define almost GF-structure on T(M) [1]. If G be Riemannian metric on M then G^C given by

$$G^C(X^C, Y^C) = G(X, Y)^C \tag{1.5}$$

for each $X, Y \in \Im_0^1$ (M) defines the Riemannian metric on T(M)

2. Induced Structures in T(M)

In this section we shall prove the following theorems.

Theorem 2.1. If *F gives an almost GF-contact structure on the basespace M then* (1,1) *tensorfield K given by*

$$K = F^C + \left(\frac{\beta^{2+1}}{a\gamma}\right)\eta^V \otimes \xi^C + \frac{\beta}{a}\eta^C \otimes \xi^H - \frac{\beta}{a}\eta^H \otimes \xi^V + \frac{\gamma}{a}\eta^C \otimes \xi^C$$

defines GF-structure an T(M), $\beta, \gamma \in R, \gamma \neq 0$.

Proof: Proof follows easily by virtue of equation (1.1) of previous section and equation on (3.26) on page 20 in [3].

Theorem 2.2. *For* (1,1) *tensorfield F admitting almost GF-contact structure on M, the* (1,1) *tensorfield L given by*

$$L = F^C + \left(\frac{\beta^{2+1}}{a\gamma}\right)\eta^V \otimes \xi^V + \frac{\beta}{a}\eta^V \otimes \xi^H - \frac{\beta}{a}\eta^H \otimes \xi^V + \frac{\gamma}{a}\eta^H \otimes \xi^H$$

gives on almost GF-structure on T(M).

Proof: Proof follows easily in view of equation (1,1) of previous section and equation on page 119 [3].

Theorem 2.3. *The* (1,1) *tensofield J defined as*

$$JX^V = aX^H, \; JX^H = aX^V \tag{2.3}$$

for each $X \in \Im_0^1(M)$ gives almost GF-structure on T(M)

Proof: We have in view of equation (2.3)

$$J^2X^V = aJX^H = a^2X^V$$

and

$$J^2X^H = aJX^V = a^2X^H$$

Hence, $J^2 = a^2 I_{2n}$ on T(M) which proves the proposition.

3. Curvature Identities

Suppose the basespace M admits the Riemannian metric G and the Riemannian connection ∇. It is well known that G^C given by

$$G^C(X^C, Y^C) = G(X, Y))^C \tag{3.1}$$

defines the Riemannian metric and ∇^C given by

(3.2) $$G^C_{X^C} Y^C = (\nabla_X Y)^C$$

defines the Riemannian connection in T(M). If R(X,Y) Z be the curvature tensor of M with respect to connection ∇.

(3.3) $$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is well known that

(3.4) $$R^C(X^C, Y^C)\, Z^C = \nabla^C_{X^C} \nabla^C_{Y^C} Z^C - \nabla^C_{Y^C} \nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C$$

is curvature tensor in T(M)

Theorem 3.1. *If* R(X,Y)Z *be the Riemannian curvature tensor for the basespace M then.*

$$R^C(X^C, Y^C)Z^C + R^C(Y^C, Z^C)X^C + R^C(Z^C, X^C)Y^C = 0 \text{ in } T(M)$$

Proof: We have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Taking complete lift, we have

$$R^C(X^C, Y^C)\, Z^C = \nabla^C_{X^C} \nabla^C_{Y^C} Z^C - \nabla^C_{Y^C} \nabla^C_{X^C} Z^C - \nabla^C_{[X^C, Y^C]} Z^C$$

Interchanging X^C, Y^C, Z^C cyclically and adding all the three equations we get

$$\begin{aligned} R^C(X^C, Y^C)\, Z^C + R^C(Y^C, Z^C)\, X^C + R^C(Z^C, X^C) Y^C &= \nabla^C_{X^C} \{\nabla^C_{Y^C} Z^C - \nabla^C_{Z^C} Y^C\} \\ &+ \nabla^C_{Y^C} \{\nabla^C_{Z^C} X^C - \nabla^C_{X^C} Z^C\} \\ &+ \nabla^C_{Z^C} \{\nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C\} \\ &- \nabla^C_{[Y^C, Z^C]} X^C - \nabla^C_{[Z^C, X^C]} Y^C - \nabla^C_{[X^C, Y^C]} Z^C \end{aligned}$$

Since $\nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C = [X^C, Y^C]$ as ∇^C is Riemannian connection in T(M) hence

$$\begin{aligned} &R^C(X^C, Y^C)Z^C + R^C(Y^C, Z^C)X^C + R^C(Z^C, X^C)Y^C \\ &= \{\nabla^C_{X^C}[Y^C, Z^C] - \nabla^C_{[Y^C, Z^C]} X^C\} + \{\nabla^C_{Y^C}[Z^C, X^C] - \nabla^C_{[Z^C, X^C]} Y^C\} \\ &\quad + \{\nabla^C_{Z^C}[X^C, Y^C] - \nabla^C_{[X^C, Y^C]} Z^C\} \\ &= [X^C, [Y^C, Z^C]] + [Y^C, [Z^C, X^C]] + [Z^C, [X^C, Y^C]] \\ &= 0 \end{aligned}$$

by Jacob Identity.

Hence the proposition.

Theorem 3.2. *If in the basespace* M *for each* $X, Y \in \Im_0^1(M)$

$$K(K, Y) = \frac{g(X, R(X, Y)Y)}{g(X, X)\, g(Y, Y) - \{g(X, Y)\}^2}$$

Then in T(M), for each real numbers s, t

$$K^C(X^C, Y^C) = K^C(sX^C, tY^C).$$

Proof: As given

$$K(K, Y) = \frac{g(X, R(X, Y)Y)}{g(X, X)\, g(Y, Y) - \{g(X, Y)\}^2}$$

Taking complete lift we obtain

$$K^C(X^C, Y^C) = \frac{g^C(X^C, R^C(X^C, Y^C)Y^C}{g^C(X^C, X^C)\, g^C(Y^C, Y^C) - \{g^C(X^C, Y^C)\}^2}$$

It is easy to show that

$$K^C(sX^C, tY^C) = K^C(X^C, Y^C).$$

REFERENCES

[1] Bejon, Cornelia-Livia, (1983): *Some Structures Induced on the Tangent Bundle of An Almost Contact Manifold.* Conferinta Nationala De Geometrie si Topologie-Platra Neamt pp. 183-186.

[2] K.L. Duggal (1971): *On Differentiable Structures Defined by Algebraic Equation* I, Nijenhuis Tensor. Tensor, N.S., Vol. 22(2), pp. 238-242.

[3] K. Yano and S. Ishihara (1973): *Tangent and Contangent Bundles*. Marcel Dekker, Inc., New York.

[4] N.J. Hicks (1964), *Notes on Differential Geometry.*, D. Van Nostrand Company, Inc. Princeton, New York.

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