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On Submanifolds Immersed In A HSU-Quaternion Manifold

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Abstract: Integrability conditions of an almost quaternion manifold were studied by Yano and Ako [4]. Quaternion submanifolds of co-dimension r have been defined and studied by Prof. A. Hamoui [1] and others. In this paper, we have defined a Hsu-quaternion manifold and showed that a submanifold of codimension r of the Hsu quaternion manifold admits Hsu-$(F, \underset{x}{U}, u^x, \eta_x^y)$-structure.

1. Preliminaries

A Hsu-quaternion manifold is the manifold M^{4n} admitting a set of tensor fields $\overset{*}{F}, \overset{*}{G}, \overset{*}{H}$ of type (1,1) satisfying following relations [2].

$$(1.1) \qquad \overset{*}{F}{}^2 = a^r I_n, \quad \overset{*}{G}{}^2 = b^r I_n$$

$$\overset{*}{H}{}^2 = c^r I_n ; \quad 0 \le r \le n \text{ and } c^r = a^r b^r$$

In being identity operator ; a, b, c complex numbers and r an integer such that

$$(1.2) \qquad \begin{aligned} &\text{(a)} \quad b^r \overset{*}{F} = \overset{*}{G}\overset{*}{H} = \overset{*}{H}\overset{*}{G} \\ &\text{(b)} \quad a^r \overset{*}{G} = \overset{*}{H}\overset{*}{F} = \overset{*}{F}\overset{*}{H} \\ &\text{(c)} \quad \overset{*}{H} = \overset{*}{F}\overset{*}{G} = \overset{*}{G}\overset{*}{F} \end{aligned}$$

2. Structure in the submanifold M^{4n-r}

Let M^{4n-r} be a submanifold of co-dimension r of the Hsu-quaternion manifold M^{4n}. Let τ denotes the immersion $M^{4n-r} \rightarrow M^{4n}$. If $B = d\tau$, then a vector field X tangent to M^{4n-r} corresponds to a vector field BX tangent to M^{4n}.

Let $\underset{x}{N}$, $x = 1, 2, \ldots, r$ be r mutually orthogonal unit normals to M^{4n-r}. The transformation for $\overset{*}{F}BX$ and $\overset{*}{F}\underset{x}{N}$ can be expressed in the form

$$\text{(2.1)} \qquad \overset{*}{F}BX = BFX + \sum_{x=1}^{r} \overset{x}{u}(x)\underset{x}{N}$$

F is a tensor field of type (1,1) and $\overset{x}{u}$ are $r(C^{\infty})$ 1- forms on submanifold M^{4n-r}.

Also,

$$\text{(2.2)} \qquad \overset{*}{F}\underset{x}{N} = -B\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}$$

$\underset{x}{U}$, $x = 1, 2, \ldots, r$ are $r(C^{\infty})$ vector fields on the submanifold M^{4n-r}

Similarly for the tensor fields $\overset{*}{G}$ and $\overset{*}{H}$ we have the following set of transformations

$$\text{(2.3)} \quad \text{(a)} \qquad \overset{*}{G}BX = BGX + \sum_{x=1}^{r} \overset{x}{v}(X)\underset{x}{N}$$

$$\text{(b)} \qquad \overset{*}{G}\underset{x}{N} = -B\underset{x}{V} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}$$

$\overset{x}{v}$ and $\underset{x}{V}$ $r(C^{\infty})$ 1-forms and vector fields respectively. G is the tensor field of type (1,1) on the submanifold M^{4n-r}.

$$\text{(2.4)} \quad \text{(a)} \qquad \overset{*}{H}BX = BHX + \sum_{x=1}^{r} \overset{x}{w}(X)\underset{x}{N}$$

$$\text{(b)} \qquad \overset{*}{H}\underset{x}{N} = -B\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}$$

$\overset{x}{w}$, $\underset{x}{W}$ and H have their usual meanings as in (2.2) and (2.3).

A manifold Vm is said to possess Hsu-$(F, \underset{x}{U}, u^{x}, \eta_x^y)$- structure if there exists tensor F of type (1,1), $r\,(C^{\infty})$ vector fields $\underset{x}{U}$, $r\,(C^{\infty})$ 1-forms $\overset{x}{u}$, $x = 1,2,\ldots,r$ and scalar functions η_x^y satisfying

$$\text{(a)} \qquad F^2 = a^r I_n + \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U}$$

$$\text{(b)} \qquad \overset{y}{u}\, oF = \sum_{x=1}^{r} \eta_x^y \, \overset{x}{u} = 0$$

$$\text{(c)} \qquad F\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \, \underset{y}{U} = 0$$

$$\text{(d)} \qquad -\overset{z}{u}(\underset{x}{U}) + \sum_{y=1}^{r} \eta_y^z \, \eta_x^y = a^r I_n$$

A manifold Vm will be to possess Hsu-$(F, G, H, \underset{x}{U}, \underset{x}{V}, \underset{x}{W}, \overset{x}{u}, \overset{x}{v}, \overset{x}{w}, \eta_x^y)$-structure if there exists tensor fields F, G, H each of type (1,1), $r\,(C^{\infty})$ vector fields $\underset{x}{U}, \underset{x}{V}, \underset{x}{W}$, and $r\,(C^{\infty})$ 1-forms $\overset{x}{u}, \overset{x}{v}, \overset{x}{w}$ and scalar functions η_x^y satisfies

$$\text{(a)} \qquad GH = b^r F + \sum_{x=1}^{r} \overset{x}{w} \otimes \underset{x}{V}$$

$$\text{(b)} \qquad \overset{y}{v}\, oH = b^r \overset{y}{u} - \sum_{x=1}^{r} \eta_x^y \, \overset{x}{w}$$

$$\text{(c)} \qquad G\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y \, \underset{y}{V} = b^r \underset{x}{U}$$

$$\text{(d)} \qquad -\overset{z}{v}(\underset{x}{W}) + \sum_{y=1}^{r} \eta_x^y \, \eta_y^z = \eta_x^z$$

$$, x, y, z = 1, 2, \ldots,$$

3. Some results

In this section, we shall prove some theorems on the submanifolds M^{4n-r}

Theorem 3.1. *A submanifold* M^{4n-r} *of co-dimension* r *of the Hsu-quaternion manifold* M^{4n} *admits* $(F, \underset{x}{U}, \overset{x}{u}, \eta_x^y)$*-structure.*

Proof: Applying $\overset{*}{F}$ to (2.1) and making use of equations (2.2) and (1.1) we get

$$a^r BX = BF^2X + \overset{y}{u}(FX)\underset{y}{N} + \sum_{x=1}^{r} \overset{x}{u}(X)\left\{-B\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}\right\}$$

Comparison of tangent and normal vector fields gives

(3.1)

$$\text{(a)} \qquad F^2 = a^r I_n + \sum_{x=1}^{r} \overset{x}{u} \otimes \underset{x}{U} \text{ and}$$

$$\text{(b)} \qquad \overset{y}{u}\, oF = \sum_{x=1}^{r} \eta_x^y \overset{x}{u} = 0$$

Again applying $\overset{*}{F}$ on (2.2) and use of (1.1), (2.1) and (2.2) we get

$$a^r \underset{x}{N} = -\left\{BF\underset{x}{U} + \sum_{z=1}^{r} \overset{z}{U}(\underset{x}{U})\underset{z}{N}\right\} + \sum_{y=1}^{r} \eta_x^y \left\{-B\underset{y}{U} + \sum_{z=1}^{r} \eta_x^z \underset{z}{N}\right\}$$

Comparing of tangential and normal vector fields we get

(3.2)

$$\text{(a)} \qquad F\underset{x}{U} + \sum_{y=1}^{r} \eta_x^y \underset{y}{U} = 0$$

$$\text{(b)} \qquad -\overset{z}{u}\left(\underset{x}{U}\right) + \sum_{y=1}^{r} \eta_y^z \eta_x^y = a^r I_n$$

In view of the equations (3.1 (a), (b)) and (3.2(a), (b)) we evidently observe that the submanifold M^{4n-r} of co-dimension r of M^{4n} admits $(F, \underset{x}{U}, \overset{x}{u}, \eta_x^y)$-structure.

Corollary (3.1)

The submanifold M^{4n-r} of co-dimension r of the Hsu-quaternion manifold M^{4n} also admits the structure $(G, \underset{x}{V}, \overset{x}{v}, \eta_x^y)$ and $(F, \underset{x}{W}, \overset{x}{w}, \eta_x^y)$ relative to tensor fields $\overset{*}{G}$ and $\overset{*}{H}$ respectively.

Theorem 3.2. *An orientable submanifold of co-dimension r of the Hsu-quaternion manifold* M^{4n} *admits,* F, G, H 3*–structures expressed* $(F, G, H, \underset{x}{U}, \underset{x}{V}, \underset{x}{W}, \overset{x}{u}, \overset{x}{v}, \overset{x}{w}, \eta_x^y)$.

Proof: From (1.2(a)) we have

$$\overset{*}{G}\,\overset{*}{H}\,BX = b^r\,\overset{*}{F}\,BX$$

which in view of (2.1) and (3.1(a)) yields

$$BGHX + \sum_{y=1}^{r} \overset{y}{v}(HX)\underset{y}{N} + \sum_{x=1}^{r} \overset{x}{w}(X)\left\{-B\underset{x}{V} + \sum_{y=1}^{r} \eta_x^y \underset{y}{N}\right\} = b^r\left\{BFX + \sum_{y=1}^{r} \overset{y}{u}(X)\underset{y}{N}\right\} \tag{3.3}$$

Comparison of tangential and normal vector fields yields

(3.4)

$$\text{(a)}\qquad GHX = b^r FX + \sum_{x=1}^{r} \overset{x}{w}(X)\underset{x}{V}$$

$$\text{(b)}\qquad b^r\,\overset{y}{u}(X) = \overset{y}{v}(HX) + \sum_{x=1}^{r} \eta_x^y\,\overset{x}{w}(X)$$

$x, y = 1, 2, \ldots, r.$

We also have from the same equation (2.2)

$$\overset{*}{G}\,\overset{*}{H}\,\underset{x}{N} = b^r\,\overset{*}{F}\,\underset{x}{N}$$

$$-\left\{BG\underset{x}{W} + \sum_{z=1}^{r} \overset{z}{v}(\underset{x}{W})\underset{z}{N}\right\} + \sum_{y=1}^{r} \eta_x^y\left\{-B\underset{y}{V} + \sum_{z=1}^{r} \eta_y^z \underset{y}{N}\right\} = b^r\left\{-B\underset{x}{U} + \sum_{z=1}^{r} \eta_x^z \underset{z}{N}\right\}$$

Equating tangential and normal vector fields we get

(3.5)

$$\text{(a)}\qquad G\underset{x}{W} + \sum_{y=1}^{r} \eta_x^y \underset{y}{V} = b^r \underset{x}{U}$$

$$\text{(b)}\qquad -\overset{z}{v}\left(\underset{x}{W}\right) + \sum_{y=1}^{r} \eta_x^y\,\eta_y^z = \eta_x^z$$

Similarly we obtain

(3.6)

$$\text{(a)}\qquad HF = a^r G + \sum_{x=1}^{r} \overset{x}{u} \otimes W_x \quad \text{etc.}$$

$$\text{(b)}\qquad FG = H + \sum_{x=1}^{r} \overset{x}{v} \otimes U_x \quad \text{etc.}$$

Further in view of the relations (1.2(a)) it follows that

$$\overset{*}{G}\overset{*}{H} BX = \overset{*}{H}\overset{*}{G} BX$$

which in view of equations (2.3(a)) and (2.4(a)) becomes

$$BGHX + \sum_{y=1}^{r} \overset{y}{v}(HX) N_y + \sum_{x=1}^{r} \overset{x}{w}(X) \left\{ -B V_x + + \sum_{y=1}^{r} \eta_x^y N_y \right\}$$

$$= BHGX + \sum_{y=1}^{r} \overset{y}{w}(GX) N_y + \sum_{x=1}^{r} \overset{x}{v}(X) \left\{ -B W_x + \sum_{y=1}^{r} \eta_x^y N_y \right\}$$

Equating tangential and normal vector fields we obtain

(3.7)

$$\text{(a)}\qquad (GH - HG)X = -\sum_{x=1}^{r} \left\{ \overset{x}{v}(X) W_x + \overset{x}{w}(X) V_x \right\}$$

$$\text{(b)}\qquad \overset{x}{v}(HX) - \overset{y}{w}(GX) = \sum_{x=1}^{r} \eta_x^y \left\{ \overset{x}{v}(X) - \overset{x}{w}(X) \right\}$$

Again

$$\overset{*}{G}\overset{*}{H} N_x = \overset{*}{H}\overset{*}{G} N_x, \qquad x = 1, 2, \ldots, r$$

$$BG W_x + \sum_{z=1}^{r} \overset{z}{v}(W) N_z + \sum_{y=1}^{r} \eta_x^y \left\{ -B V_y + \sum_{z=1}^{r} \eta_y^z N_z \right\}$$

$$= BH V_x + \sum_{z=1}^{r} \overset{z}{w}(V)_x N_z + \sum_{y=1}^{r} \eta_x^y \left\{ -B W_y + \sum_{z=1}^{r} \eta_y^z N_z \right\}$$

Equating tangential and normal vector fields we get

$$(3.8)\qquad \begin{aligned} &(a)\qquad G\underset{x}{W} - H\underset{x}{V} = -\sum_{y=1}^{r} \eta_x^y \left(\underset{y}{W} - \underset{y}{V}\right) \\ &(b)\qquad \overset{z}{v}(\underset{x}{W}) - \overset{z}{w}(\underset{x}{v}) = 0 \end{aligned}$$

for $x, y, z = 1, 2, \ldots, r.$

In a similar manner, we can prove the following of relations

$$(3.9)\qquad \begin{aligned} &(a)\qquad (HF-FH)X = -\sum_{x=1}^{r} \left\{ \overset{x}{v}(X)\underset{x}{U} + \overset{x}{u}(X)\underset{x}{V} \right\} \\ &(b)\qquad \overset{y}{w}(FX) - \overset{y}{u}(HX) = \sum_{y=1}^{r} \eta_x^y \left\{ \overset{y}{w}(X) - \overset{y}{u}(X) \right\} \\ &(c)\qquad H\underset{x}{U} - F\underset{x}{W} = \sum_{y=1}^{r} \eta_x^y \left(\underset{y}{U} - \underset{y}{W}\right) \\ &(d)\qquad \overset{z}{w}(\underset{x}{U}) - \overset{z}{w}(\underset{x}{V}) = 0 \quad \text{etc.} \end{aligned}$$

The theorem follows by virtue of equation (3.5 to (3.9).

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