The Nepali Math. Sc. Report
Vol. 29, No. 1 & 2, 2009

On the Approximation of Conjugate of Functions Belonging To Lip $\{\xi(t), p\}$ Class By Generalized Nörlund Means

SHYAM LAL
JITENDRA KUMAR KUSHWAHA

Abstract: In this paper, the degree of approximation of conjugate of functions belonging to Lip $\{\xi(t), p\}$ class by generalized Nörlund means of conjugate series of **Key words:** model, indirect technique, ratio, parameters, mortality, deaths. Fourier series has been determined.

1. INTRODUCTION AND DEFINITION

Qureshi ([6]) has determined the degree of approximation of function $\tilde{f}(x)$, conjugate of a function $f \in \text{Lip}\alpha$, $\text{Lip}(\alpha, p)$ by Nörlund method .The purpose of this paper is to generalize above result in two ways and to determine the approximation of $\tilde{f}(x)$, conjugate of a function $f \in \text{Lip}\{\xi(t), p\}$ class, by generalized Nörlund means .

Let f be periodic with period 2π and integrable over$(-\pi, \pi)$ in Lebesgue sense. Let its Fourier series be given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x) . \qquad (1)$$

The conjugate series of the Fourier series (1) is given by

$$\sum_{n=1}^{\infty}\left(a_n \sin nx - b_n \cos nx\right) = -\sum_{n=1}^{\infty} B_n(x). \qquad (2)$$

We define norm $\| \|_p$ by $\| f \|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}}$, $p \geq 1$

and the degree of approximation $E_n(f)$ is given by (Zygmund [8])

$$E_n(f) = \min \| f - T_n \|_p$$

where $T_n(x)$ is a trigonometric polynomial of degree n.

A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^{\alpha})$, for $0 < \alpha \leq 1$.

$f(x) \in Lip(\alpha, p)$ for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p \, dx \right)^{1/p} = O(|t|^{\alpha}), \quad 0 < \alpha \leq 1 \qquad \text{(McFadden [5])}.$$

Given a positive increasing function $\xi(t)$ and an integer $p \geq 1$,

Keywords and phrases : Lip $\{\xi(t), p\}$ class of functions , Fourier series , Degree of approximation . Generalized Nörlund means.
Subject of classification (2007) : 42B05 , 42B08.

$f(x) \in Lip(\xi(t), p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(t)|^p \, dx \right)^{\frac{1}{p}} = O(\xi(t)) \qquad \text{(Siddiqi [7])}. \qquad (3)$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having its n^{th} partial sum $s_n = \sum_{v=0}^{n} u_n$.

Let $\{p_n\}$ and $\{q_n\}$ be two sequences of real numbers such that

$$R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ \forall n \geq 0 .$$

For any sequence $\{s_n\}$ we write

$$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} s_{n-k}\,. \tag{4}$$

The generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$. If $t_n^{p,q} \to s$, as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable S by generalized Nörlund method (N,p,q) and is denoted by $S_n \to S(N,p,q)$.

(Borwein [1])

The necessary and sufficient conditions for a (N, p, q) method to be regular are

$$\sum_{k=0}^{n} |p_{n-k} q_k| = O(|R_n|)$$

and $p_{n-k} = o(|R_n|)$, as $n \to \infty$, for every fixed $k \geq 0$ for which $q_k \neq 0$.

The (N, p, q) method reduces to the Nörlund method if $q_n = 1$ for all n. The method (N, p, q) reduces to Riesz method $(\tilde{N}, q_n)$ if $p_n = 1$, for all n .When $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, and $q_n = 1\ \forall\, n$, the method (N,p,q) reduces to (C, α).

We use following notations:

$$\psi(t) = f(x+t) - f(x-t), \qquad \tilde{f}(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi(t)\cot\tfrac{t}{2}\,dt$$

2. MAIN THEOREM

We prove the following:

Theorem: Let the regular generalized Nörlund method (N, p, q) be defined by a non- negative, monotonic non-increasing sequence $\{p_n\}$ and a non-negative, monotonic non- decreasing sequence $\{q_n\}$ of real constants such that

$$q_n p_n = O(R_n \log n) \text{ with } n \geq n_0 > 1\,. \tag{5}$$

If $f : R \to R$ is a 2π periodic, Lebesgue integrable and belonging to Lip($\xi(t)$,p) class, $\xi(t)$ is positive increasing function of t satisfying

$$\left\{ \int_0^{\frac{1}{n}} \left(\frac{t\,|\psi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left(\frac{1}{n}\right) \tag{6}$$

and

$$\left\{ \int_{\frac{1}{n}}^{\pi} \left(\frac{t^{-\delta}\,|\psi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left(n^{\delta}\right) \tag{7}$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, q the conjugate index of p and the condition (6) and (7) hold uniformly in x, then degree of approximation of $\tilde{f}(x)$, conjugate of $f \in Lip\{\xi(t), p\}$, by generalized Nörlund means

$\tilde{t}_n^{\,p,q}(x) = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} \tilde{s}_k$ of the conjugate series (2) is given by

$$\left\| \tilde{t}_n^{\,p,q}(x) - \tilde{f}(x) \right\|_p = O\left(n^{\frac{1}{p}}\, \xi\left(\frac{1}{n}\right) \log n \right) \tag{8}$$

3. LEMMAS

The following lemmas are required for the proof of our theorem

***Lemma* 1** (McFadden , 1942), If $\{p_n\}$ is a non-negative non-increasing sequence for $0 \le a \le b \le n$, $0 < t \le \pi$ then

$$\left| \sum_{k=0}^{n} \frac{p_k \cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right| = O\left(\frac{p_\tau}{t}\right)$$

***Lemma* 2** If $\{p_n\}$ is a non-negative non-increasing and $\{q_n\}$ is a non-negative non- decreasing sequence then

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| = O\left(\frac{q_n p_\tau}{R_n t}\right) \qquad \text{if} \quad \frac{1}{n}\le t\le\pi$$

Proof By Abel's lemma, we have

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n}\frac{p_k q_{n-k}\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \le \frac{q_n}{\pi R_n}\max_{0\le m\le n}\left|\sum_{k=0}^{n}\frac{p_k\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|$$

$$= O\left(\frac{q_n p_\tau}{R_n t}\right), \text{ by Lemma 1 .}$$

4. PROOF OF THE THEOREM

The n^{th} partial sum of conjugate Fourier series is given by

$$\tilde{S}_n(x) = -\frac{1}{2\pi}\int_0^\pi \cot\frac{t}{2}\,\psi(t)dt + \frac{1}{2\pi}\int_0^\pi \frac{\cos\left(n-k\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt$$

$$\tilde{S}_n(x) - \left(-\frac{1}{2\pi}\int_0^\pi \cot\frac{t}{2}\,\psi(t)dt\right) = \frac{1}{2\pi}\int_0^\pi \frac{\cos\left(n-k\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt\,.$$

By taking (N, p, q) means of $\tilde{S}_n(x)$, we get

$$\bar{t}_n^{\,p,q}(x)-\bar{f}(x)=\frac{1}{2\pi R_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}p_k q_{n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$=\frac{1}{2\pi R_n}\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right)\psi(t)\sum_{k=0}^{n}p_k q_{n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$$=I_1+I_2. \qquad (9)$$

Applying Hölder's inequality and the fact that $\psi(t)\in Lip(\xi(t),p)$, we have

$$|I_1|\le\frac{1}{2\pi R_n}\left\{\int_0^{\frac{1}{n}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n}}\left(\frac{\xi(t)}{t}\sum_{k=0}^{n}p_k q_{n-k}\left|\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\right)^q dt\right\}^{\frac{1}{q}}$$

$$\le O\left(\frac{1}{n}\right)\left\{\int_0^{\frac{1}{n}}\left(\frac{\xi(t)}{t^2}\right)^q dt\right\}^{\frac{1}{q}}, \quad \text{by (6)}$$

$$=O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left(\int_{\epsilon}^{\frac{1}{n}}\frac{1}{t^{2q}}dt\right)^{\frac{1}{q}}, \text{ for some } 0<\epsilon<\tfrac{1}{n}$$

by second mean value theorem for integral.

$$=O\left(n^{\frac{1}{p}}\xi\left(\frac{1}{n}\right)\right) \qquad (10)$$

Similarly, as above, we have

$$|I_2|=\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)}{2\pi R_n t^{-\delta}}\sum_{k=0}^{n}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(n^{\delta}\right)\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)q_n\,p_\tau}{t^{-\delta}t\,R_n}\right)^q dt\right]^{\frac{1}{q}} \quad \text{, by (7) and Lemma 2}$$

$$= O\left(n^{\delta}\right)O\left(\frac{q_n}{R_n}\right)\left[\int_{\frac{1}{n}}^{\pi}\left(\frac{\xi(t)\,p_\tau}{t^{1-\delta}}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(\frac{n^{\delta}q_n}{R_n}\right)\left[\int_{\frac{1}{\pi}}^{n}\left\{\frac{\xi\left(\frac{1}{y}\right)p_{[y]}}{\left(\frac{1}{y}\right)^{1-\delta}}\right\}^q \frac{dy}{y^2}\right]^{\frac{1}{q}} \quad \text{, taking } t=\tfrac{1}{y}$$

$$= O\left(\frac{n^{\delta}q_n p_n}{R_n}\xi\left(\frac{1}{n}\right)\right)\left[\int_{\frac{1}{\pi}}^{n} y^{(1-\delta)q-2}\,dy\right]^{\frac{1}{q}} \quad \text{, by mean value theorem}$$

$$= O\left(\frac{n^{\delta}q_n p_n}{R_n}\xi\left(\frac{1}{n}\right)\right)\left(\frac{n^{q(1-\delta)-1}-\left(\frac{1}{\pi}\right)^{q(1-\delta)-1}}{q(1-\delta)-1}\right)^{\frac{1}{q}}$$

$$= O\left(n^{1/p}\xi\left(\tfrac{1}{n}\right)\log n\right), \quad \text{by (5) and hypothesis of theorem} \qquad (11)$$

Combining from (9) to (11), we have

$$\left\|\tilde{t}_n^{\,p,q}(x)-\tilde{f}(x)\right\|_p = O\left(n^{1/p}\,\xi\left(\tfrac{1}{n}\right)\log n\right)$$

5. COROLLARIES

Following Corollaries can be derived from the theorem.

***Corollary* 1** If $\xi(t)=t^{\alpha}$ then the degree of approximation of a function belonging to Lipα, $\frac{1}{p}<\alpha<1$ is given by

$$\left\| \tilde{t}_n^{p,q} - \tilde{f} \right\| = O\left(\frac{\log n}{n^{\alpha - \frac{1}{p}}} \right)$$

***Corollary* 2** If $p \to \infty$ in Cor.1 then we have for, $0 < \alpha < 1$,

$$\left\| \tilde{t}_n^{p,q}(x) - \tilde{f}(x) \right\| = O\left(\frac{\log n}{n^{\alpha}} \right)$$

***Remark*:-**An independent proof of Corollaries (1) and (2) can be developed along the same lines as the theorem.

REFERENCES

Borwein, D., On products of sequences, *J. London Math. Soc.*, **33** (1958), 352-357.

Hardy, G. H., On the summability of Fourier series, *Proc. London Math. Soc.*, **12**(1913), 365-372.

Khare, S. P., Generalized Nörlund summability of Fourier series and its conjugate series, *Indian J. Pure Appl. Math.* **21**(1990) no. 5, 457-467.

Khan, Huzoor H., On the degree of approximation of functions belonging to the class Lip (α, p), *Indian J. Pure Appl. Math.*, **5**(1974) no.2, 132-136.

McFadden, Leonard, Absolute Nörlund summability, *Duke Math. J.*, **9**(1942), 168-207.

Qureshi, K., On the degree of approximation of functions belonging to the class Lip(α, p) by means of a conjugate series, *Indian J. Pure Appl. Math.*, **13**(1882) no.5, 560-563.

Siddiqi, A. H., Ph.D. Thesis (1967), *Aligarh Muslim University, Aligarh.*

Zygmund, A., Trigonometric series, Vol. I , II, Second edition (1959), *Cambridge University Press.*