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On Weakly Symmetric and Weakly Ricci-Symmetric Lorentzian Para-Sasakian Manifolds

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Abstract: The aim of the present paper is to study weakly symmetric and weakly Ricci symmetric an LP-Sasakian manifolds and proved that if weakly symmetric LP-Sasakian manifold satisfies η-parallel Ricci condition then the scalar curvature of the manifolds is equal to rank of ϕ. If weakly symmetric an LP-Sasakian manifolds satisfies Ricci tensor of Coddazi type then the manifolds is R-Harmonic, further we have obtained the necessary condition for weakly symmetric an LP-Sasakian manifolds has Cyclic Ricci is that vanishing the 1-forms $\alpha + \beta + \delta$ for all the vector field X, Y and Z.

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Key words: Coddazi type Ricci tensor, R-harmonic manifold, CyclicRicci tensor, η-parallel Ricci tensor, and Quasi Einstein manifold.

1. Introduction

K. Matsumoto 1989, Introduced the notion of Lorentzian Para-Sasakian manifold $(M^n g)$ $(n \geq 2)$ Mihai and Rosca [2] define same notion independently. This type of manifolds is also discussed in [3]. A non-flat Riemannian manifolds if called weakly symmetric if there exist 1-form α, β, δ and σ such that the relation.

$$(1.1)\quad (\nabla_X R)(Y, Z, U, V) = \alpha(X)R(Y, Z, U, V) + \beta(Y)R(X, Z, U, V) + \gamma(Z)R(Y, X, U, V) \ldots \delta(U)R(Y, Z, X, V) + \sigma(V)R(Y, Z, U, V)$$

Holds for all vector fields X, Y, Z… $\in (M^n, g)$ ([4][5])

Where R is the curvature tensor of the manifold (M^n, g) of type (0, 4)

A weakly symmetric manifolds is said to be proper if $\alpha = \beta = \gamma = \delta = \sigma = 0$ is not the case.

A Riemannian manifolds is called weakly Ricci symmetric if there exist 1-form ρ, μ and ν such that the relation

(1.2) $(\nabla_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(X, Z) + \nu(Z)S(X, Y)$

Hold for vector fields A, Y and Z where S is the Ricci tensor of type (0, 2) of the manifold.

A weakly Ricci symmetric manifolds is said to be proper if $\rho = \mu = \nu = 0$ is not the cases.

Let $[e_I]$, I = 1, 2, 3, 4…, n be an orthonarmal basis of the tangent space at a point of the manifolds.

Putting $Y = V = e_I$ in (1.1) and taking summation over i, $1 \leq i \leq n$, we get

(1.3) $(\nabla_X S)(Z, U) = \alpha(X)S(Z, U) + \beta(Z)S(X, U) + \delta(U)S(Z, X) + \beta(R(X, Z)U) + \delta(R(X, U)Z)$

L. Tamassy and T. Q. Binh ([4], [5]) introduced the notion of weakly symmetric and weakly Ricci symmetric, Sasakian manifolds. M. Kon [6], U. C. De, T. Q. Binh and A.A Shaikh [7] obtained the necessary condition for the compatibility of several k-Contact structure with weak symmetric and weak Ricci symmetric and provided they do not reduce to conman local symmetric and Pseudo Ricci symmetric respectively. In this paper we study LP–Sasakian manifolds with weakly symmetric and weakly Ricci symmetric tensor, Quasi Einstein, Cyclic Ricci tensor and Ricci tensor of Codaggi type.

2. Preliminaries

Let (M^n, g) be n-dimensional. Differentiable manifolds with a tensor filed f of type (1, 1), a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type (0, 2) which satisfying

(2. 1) $\phi^2 = 1 + \eta \otimes \xi$

(2. 2) $\eta\xi = -1$

(2.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$

(2.4) $g(X, \xi) = \eta(X)$

It is called Lorentzian Para-Sasakian manifolds briefly (LP-Sasakian manifold) and the structure (ϕ, η, ξ, g) is called a Lorentzian Para-contact structure. An LP-Contact manifolds is called LP-Sasakian manifold if it satisfying.

(2.5) $(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi$

Where ∇ denoted the operator of covariant differention with respect to Lorentzian g metric of type (0, 2).

In a LP-Sasakian manifolds the following relations hold ([1][2]).

(2.6) (a) $\phi\xi = 0$. (b) $\eta(\phi X) = 0$ (c) rank $(\phi) = \eta - 1$

(2.7) $(\nabla_X \xi = \phi X$

(2.8) $\eta(R(\xi, X)Y = g(X, Y) - \eta(Y)\eta(X)$

(2.9) $R(\xi, X)\xi = X + \eta(X)\xi$

(2.10) $S(X, \xi) = (n-1)\eta(X)$

(2.11) $R(X, Y).\xi = \eta(Y)X - \eta(X)Y$

(2.12) $S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y)$

For any vector field X, Y and Z where $R^*(X, Y)Z$ is the Riemannian curvature tensor.

Lemma 2.1:

In a Lorentzian Para-Sasakian manifolds if ξ be a killing vector fields then we get.

(a) $L_\xi S = 0$

(b) $L_\xi r = 0$, where L denoted Li-derivation

3. Weakly symmetric and weakly Ricci symmetric LP-Sasakian manifolds with $\nabla_\xi S)(Z, U) = 0$

Theorem 3.1:

If weakly symmetric and weakly Ricci symmetric LP-Sasakian manifolds (M^n, g) $n \geq 3$ satisfies the condition then the sum of the 1-form $\alpha + \beta + \delta$ are equal to the sum of the 1-form $\rho + \mu + \nu$ over the killing vector field ξ.

Proof:

Consider the manifolds (M^n, g) is weakly symmetric. Then form (1.3) putting $X = \xi$ and using (2.10), we get

(3.1) $(\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + (n-1)[\beta(Z)\eta(U) + \delta(U)\eta(Z)] + \beta(P(\xi, Z)U) + \delta(R(\xi, U)Z)$

From (2.1-a), we get

(3.2) $(\nabla_\xi S)(Z, U) = -S(\nabla_Z\xi, U) - S(Z, \nabla_U\xi)$

Using (2.7) in (3.2), we get

(3.3) $(\nabla_\xi S)(Z, U) = -S(\phi Z, U) + S(Z, \phi U)$

Also from (2.11) we have

$S(\phi Z, U) = ng(\phi Z, U) - g(\phi Z, U)$ and

$S(Z, \phi U) = ng(Z, \phi U) - g(Z, \phi U)$

Using this result in (3.3), we get

(3.4) $(\nabla_\xi S)(Z, U) = 0$

From (1.2) We get

(3.5) $(\nabla_\xi S)(Z, U) = \rho(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \nu(U)S(\xi, Z)$

Also from (3.1) we get

(3.6) $(\nabla_\xi S)(Z, U) = \alpha(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \delta(U)S(\xi, Z) + \beta(R(\xi, U)Z)U) + \delta(R(\xi, U)Z)$

From (3.4), (3.5) and (3.6), we get

(3.7) $\alpha(\xi)S(Z, U) + \beta(Z)S(\xi, U) + \delta(U)S(\xi, Z) + \beta(R(\xi, U)Z)U) + \delta(P(\xi, U)Z) = \rho(\xi)S(Z, U) + \mu(Z)S(\xi, U) + \nu(U)S(\xi, Z)$

Putting $Z = U = \xi$ in (3.7), we get

$$\alpha(\xi) + \beta(\xi) + \delta(\xi) = \rho(\xi) + \mu(\xi) + \nu(\xi)$$

We get the result as required.

4. Weakly Symmetric Quasi-Einstein an LP-Sasakian Manifolds

Definition 4.1 A non-flat LP-Sasakian manifold is called Quasi-Einstein manifold if its Ricci tensor S of type (0, 2) satisfies the condition

(4.1) $S(X, Y) = ag(X, Y) + bA(X)A(Y)$

Where a, b are scalar and $b \neq 0$ and A is non-zero 1-form such that $A(X) = g(X, U)$ for every X and U is unit vector fields.

Theorem 4.10:

There exist no weakly symmetric Quasi Einstein LP-Sasakian manifolds if

$r\alpha + 2\alpha\beta + 2\beta\delta + 2bA(L)U + 2bA(M)U - b||U||^2$ is not everywhere zero.

Proof:

We consider a weakly symmetric manifold is a Quasi Einstein LP-Sasakian manifold (M^n, g)

Then from (1.3), we get

(4.2) $(\nabla_X S)(Z, U) = \alpha(X)S(X, Y) + \beta(Y)S(X, Y) + \delta(Z)S(X, Y) + \beta(R(X, Y)Z) + \delta(R(X, Z)Y)$

Where $g(X, L) = \beta(X)$ and $g(X, M) = \delta(X)$

From (4.1) we get

(4.3) $(\nabla_X S)(Y, Z) = b[\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$

Then from (4.2) and (4.3), we get

(4.4) $\alpha(X)S(X, Y) + \beta(Y)S(X, Y) + \delta(Z)S(X, Y) + \beta(R(X, Y)Z) + \delta(R(X, Z)Y) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$

Putting $Y = Z = e_i$ in (4.4) and taking summation $1 \leq i \leq n$, we get

(4.5) $r(X) + 2S(X, L) + 2S(X, M) = 2b \sum_{i=1}^{n} (\nabla_X A)(e_i)A(e_i)$

Using (4.1) in (4.5), we get

(4.6) $r\alpha(X) + 2[ag(X, L) + bA(X)A(L)] + 2[Ag(X, M) + bA(X)A(M)] = 2b \sum_{i=1}^{n} (\nabla_X A)(e_1)A(e_1)$

The right hand side of equation (4.6) can be written as follows

$$\sum_{i=1}^{n} X(A(e_1) - A(\nabla_X e_1)]A(e_1)$$

Where $\{e_1\}$ be the orthonormal bases at $T_u(M)$, $u \in M$ let us translate these parallel from u in any direction then $(\nabla_X e_1) = 0$ using this fact in above relation, we get

(4.7) $2b \sum_{i=1}^{n} X(A(e_1))A(e_1) = 2b \sum_{i=1}^{n} g(\nabla_X U, e_1)g(e_1, U) = 2bg(\nabla_X U, U) = bXg(U, U) = bX||U||^2$

Then from (4.6) and (4.7) we get

$$R\alpha + 2\alpha\beta + 2\beta\delta + 2bA(L)U + 2bA(M)U - b||U||^2$$

We get the result as required.

5. Weakly Ricci-Symmetric and LP-Sasakian Manifolds with η-Parallel Ricci tensor.

Definition 5.1: The Ricci tensor of weakly Ricci symmetric LP-Sasakian manifolds is called η-parallel if it satisfies.

(5.1) $(\nabla_X S)(\phi X, \phi Y) = 0$, for all vector field X, Y and Z.

Theorem 5.1: If weakly Ricci symmetric LP-Sasakian manifolds has η-parallel Ricci tensor then the scalar curvature r of the manifolds is equal to rank (ϕ).

Proof:

We consider weakly Ricci symmetric LP-Sasakian manifolds with η-parallel Ricci tensor. Then from (1.2) we get

(5.2) $(\nabla_X S)(X, Y) = M(X)S(\phi X, \phi Z) + \mu(\phi Y)S(X, \phi Z) + v(\phi Y)S(X, \phi Y)$

Using (2.12) in (5.2) we get

(5.3) $(\nabla_X S)(\phi Y, \phi Z) = M(X)S(Y, Z) + (n - 1)\eta(Y)\eta(Z) + M(\phi Y)S(X, \phi Y)$

From (5.1) and (5.3), we get.

(5.4) $M(X)[S(Y, Z) + (n - 1)\eta(Y)\eta(Z)] + \mu(\phi Y)S(X, \phi Z) + \nu(\phi Z)S(X, \phi Y)$

Putting $Z = \xi$ in (5.4) and using (2.6-b) and (2.10), we get.

$$r = n - 1 = \text{rank } (f)$$

We get the result as required.

6. Weakly Symmetric an LP-Sasakian Manifolds with Cyclic Ricci Tensor

Theorem 6.1: The necessary condition for LP-Sasakian manifolds has Cyclic Ricci tensor if vanishing the sum of 1-form $\alpha + \beta + \delta$ for all vector fields S, Y and Z.

Proof:

From (1.3), we get,

(6.1) $(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \beta(Y)S(X, Z) + \delta(Z)S(Y, X) + \beta(R(X, Y)Z) + \delta(R(X, Y)Z)$

Now taking the cyclic permutation of equation (6.1) in X, Y, Z and adding them, we get.

(6.2) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = S(X, Y)[\alpha(X) + \beta(X) + \delta(X)] + S(X, Z)[\alpha(Y) + \beta(Y) + \delta(Y)] + S(Y, X)[\alpha(Z) + \beta(Z) + \delta(Z)] + \beta(R(X, Y)Z) + \beta(R(Y, Z)X) + \beta(R(Z, X)Y) + \delta(R(X, Z)Y) + \delta(R(Y, X)Z) + \delta(R(Z, Y)X)$

Using (2.8) in equation (6.2), we get

(6.3) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = S(X, Y)[\alpha(X) + \beta(X) + \delta(X)] + S(X, Z)[\alpha(Y) + \beta(Y) + \delta(Y)] + S(Y, X)[\alpha(Z) + \beta(Z) + \delta(Z)]$

From (6.3) it follows that

(6.4) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0$

If vanishing the sum of 1-form $\alpha + \beta + \delta$ for all vector fields X, Y and Z.

Corollary 6.1: If weakly symmetric KLP-Sasakian manifold (M^n, g), $n \geq 3$ satisfies the Cyclic Ricci tensor condition then the vanishing the sum of 1-form $\alpha + \beta + \delta$ over the killing vector fieldsξ.

Theorem 6.2: If weakly symmetric LP-Sasakian manifolds has Ricci tensor of Coddazzi type then the manifolds is R-Harmonic. (that is symmetric)

Proof:

We suppose weakly Ricci symmetric LP-Sasakian manifold has Ricci tensor of Coddazzi type that is LP Sasakian manifolds with the condition (6.7)

(6.7) $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) + (\nabla_X S)(X, Y)$

From equation (6.7) and equation (6.4), we get $(\nabla_X S)(Y, Z) = 0$

We get the result as required.

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