The Nepali Math. Sc. Report
Vol. 31, No. 1 & 2, 2011

Optimality of the Cyclic Sequence on Bottleneck Product Rate Variation Problem with a General Objective

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Abstract: The bottleneck product rate variation problem minimizes the maximum variation in the rate at which different models are produced. The problem has mathematically interesting base model with theoretical value and real world applications. In this paper, cyclic sequence to the problem has been shown to be optimal. Cyclic sequence reduces the time complexity.

Keywords: Product rate variation problem; sequencing problem; nonlinear integer programming

1. Introduction

The product rate variation problem (PRVP) minimizes the variation in the rate at which different models of a common base product are produced on the assembly lines [6]. The problem minimizes both the earliness and the tardiness penalties that respond to the customer demands for a variety of models without holding large inventories or incurring large shortages. This is a problem of finding a sequence of different models distributed as evenly as possible on the assembly lines with the assumption of negligible switch-over cost and unit processing time for each copy of each model.

The problem has been formulated as a non-linear integer programming with the objective of minimizing the deviation between the actual and the ideal production under the assumption that the system has sufficient capacity with negligible changeover costs from one model to another and each model is produced in a unit time [9, 10]. The problem has mathematically interesting base model with theoretical value and real world applications, see [2].

The problem has been extensively studied and solved in pseudo-polynomial time. The bottleneck PRVP i.e. the problem with the objective of minimizing the maximum deviation between the actual and the ideal production has been solved in O(DlogD) time [11, 3, 4] . Solution with pseudo-polynomial time may be expensive since this time depends on the size of the demands. Existence of cyclic sequences substantially reduces the time.

In this paper, cyclic sequence of the bottleneck product rate variation problem with a general objective is shown to be optimal.

The plan of the paper is as follows. Section 2 reviews the mathematical model. Section 3 describes the solution procedure. Section 4 shows that the cyclic sequence is optimal. The last section concludes the paper.

2. Mathematical Models

Given $d_i \in N$ demand for a model i, $i = 1,...,n$, N being the set of positive integers, with total demand $D = \sum_{i=1}^{n} d_i$ and demand ratio $r_i = \frac{d_i}{D}$, let the time horizon be partitioned into D equal units and each product is produced in a unit time. There will be k complete units of various products during the first k, $k = 1,...,D$ time units. Let x_{ik} be the quantity of product i produced during the time units 1 through k. Consider f_i, $i = 1,...,n$, unimodal symmetric convex function with minimum 0 at 0.

The mathematical model of the bottleneck PRVP [7, 8] is

(1) $\quad \min \max f_i(x_{ik} - kr_i)$

subject to

(1.1) $\quad \sum_{i=1}^{n} x_{ik} = k \qquad k = 1,...,D$

(1.2) $x_{i(k-1)} \le x_{ik}$ $i = 1,...,n$; $k = 2,..., D$

(1.3) $x_{iD} = d_i$; $x_{i0} = 0$ $i = 1,...,n$

(1.4) $x_{ik} \ge 0$, integer

Constraint (1.1). shows the cumulative production during the time units 1 through k. Constraint (1.2) ensures that the total production of every product over k time units is a non-decreasing function of k. Constraint (1.3) guarantees that the demands for each product are met exactly. Constraint (1.2) and (1.4) ensure that exactly one unit of a product is scheduled during one time unit. In this paper, we consider a general objective function $f_i(x_{ik} - kr_i) = |x_{ik} - kr_i|^m$, m being a positive integer.

2. Solution procedure

The perfect matching with a bisection search, appeared in [11] for the bottleneck product rate variation problem with absolute-deviation objective, can also be applied for the Problem with necessary modifications.

The method relies on the level curves $f_{ij}(k) = |j - kr_i|^m$, $j = 0,1,...,d_i$, ; $i = 0,1,...,n; k = 1,..., D$ and the bottleneck (bound) $B > 0$. The time horizon is assumed to be continuous though is partitioned into D equal time-buckets i.e. $T = [1, D]$. A model (i, j) is sequenced in a time-bucket $k \in [1, D]$ such that the level curves do not exceed B. This introduces the earliest sequencing time $E_m(i, j)$ and the latest sequencing time $L_m(i, j)$ for (i, j), for all i, j.

For a given B, $E_m(i, j)$ and $L_m(i, j)$, $i = 0,1,...,n$; $j = 1,...,d_i$ are the unique integers $E_m(i, j) = \left\lceil \frac{j - \sqrt[m]{B}}{r_i} \right\rceil$ and $L_m(i, j) = \left\lfloor \frac{j - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor$, [5].

The earliest sequencing time $E_m(i, j)$ and the latest sequencing time $L_m(i, j)$ form a time window $T_m = [E_m(i, j), L_m(i, j)]$ within which $(i, j), i = 0,1,...,n; j = 1,...,d_i$, can be sequenced with the level curves not exceeding the bottleneck.

A V_1-convex bipartite graph $G = (V_1, V_2, E)$ is constructed sequencing (i, j) within T_m, where $V_1 = \{1,..., D\}$ stands for the set of sequencing models, $V_2 = \{(i, j) \mid i = 0,1,...,n; j = 1,...,d_i\}$ the set of (i, j) and $E = \{(k, (i, j)) \mid k \in T_m\}$.

The earliest due date (EDD) algorithm that matches each $k \in V_1$ to the unmatched (i, j) with the smallest $L_m(i, j)$ and $(k,(i,j)) \in E$ finds a perfect matching. The algorithm stops if no such (i, j) exists [11]. The perfect matching is order-preserving. An order-preserving perfect matching in G is analogous to a feasible solution to the Problem [11].

A perfect matching in G exists if and only if $|N(K)| \geq |K|$,where $N(K) = \{(i, j) : (i, j) \in V_2, \exists k \in K s.t. (k,(i, j)) \in E\}$ and K is either an interval in V_1 or the neighborhood of an interval in V_1, [1]. This is the Hall's theorem for the existence of a perfect matching that yields a feasible solution to the problem. Existence of a perfect matching depends on B. A perfect matching exists if B satisfies the inequalities in the following theorem. This is a certificate for the existence of a feasible solution.

Theorem 1 [5] *Problem F_m has a feasible solution if and only if, for all $k_1, k_2 \in V_1$ with $k_1 \leq k_2$ and $[E_m(i, j), L_m(i, j)] \cap [k_1, k_2] \neq \phi$, B satisfies the inequalities* $\sum_{i=1}^{n} (\lfloor k_2 r_i + \sqrt[m]{B} \rfloor - \lceil (k_1 - 1) r_i - \sqrt[m]{B} \rceil) \geq k_2 - k_1 + 1$ *and* $\sum_{i=1}^{n} (\lceil (k_2 r_i - \sqrt[m]{B} \rceil - \lfloor k_1 - 1) r_i + \sqrt[m]{B} \rfloor) \leq k_2 - k_1 + 1$.

A feasible solution with a minimum B is optimal. The minimum B can be obtained using a bisection search that runs between the lower and upper bottlenecks. The lower and upper bottlenecks for the problem are $(1 - r_{\max})^m$ and $(1 - \frac{1}{D})^m$, respectively.

Theorem 2 [5] *A bisection search in the interval $[(1 - r_{\max})^m, (1 - \frac{1}{D})^m]$ determines the minimum B in $O(\log D)$ time.*

The time complexity to yield an optimal sequence using the bisection search is O(DlogD) since $E_m(i,j)$ and $L_m(i,j)$ can be calculated in O(D).

3. Optimality of Cyclic Sequence

The time complexity can substantially be reduced when cyclic sequence exists. When $u=\gcd(d_1,\ldots,d_n) > 1$, cyclic sequence consisting of u subsequences with the same length exists. Furthermore, cyclic sequence is optimal.

Lemma 1 If jth copy of a model i, i = 1,…, n, is not sequenced within $[E_m(i,j),L_m(i,j)]$, the level curves exceed B.

Proof:

Suppose that j^{th} copy of a model i, $i = 1, \ldots, n$, be sequenced such that $k < E_m(i,j)$.

$\Rightarrow k < \left\lceil \frac{j-\sqrt[m]{B}}{r_i} \right\rceil$

$\Rightarrow k < \frac{j-\sqrt[m]{B}}{r_i} \Rightarrow$ if $\frac{j-\sqrt[m]{B}}{r_i}$ is not an integer.

If $\frac{j-\sqrt[m]{B}}{r_i}$ is an integer, $k = E_m(i,j)$

$\Rightarrow \sqrt[m]{B} < j-kr_i$

$\Rightarrow B < |j-kr_i|^m$

Further, suppose that $L_m(i,j) < k$.

$\Rightarrow \left\lfloor \frac{j-1+\sqrt[m]{B}}{r_i} + 1 \right\rfloor < k$

$\Rightarrow \frac{j-1+\sqrt[m]{B}}{r_i} + 1 < k$ if $\frac{j-1+\sqrt[m]{B}}{r_i} + 1$ is not an integer.

If $\frac{j-1+\sqrt[m]{B}}{r_i} + 1$ is an integer, $k = L_m(i,j)$.

$\Rightarrow \sqrt[m]{B} < (k-1)r_i(j-1)$

$\Rightarrow B < |(j-1)-(k-1)r_i|^m$.

Theorem 3 If $u = \gcd(d_1, \ldots, d_n) > 1$, a cyclic sequence to the problem, consisting

of u repetition of optimal subsequence, exists and is optimal.

Proof:

For a feasible solution, j^{th} copy of of model I, $I = 1, \ldots, n$, must be sequenced within $[E_m(i.j), L_m(i,j)]$, $I = 1, \ldots, n$; $j = 1, \ldots, d_i$.

Otherwise, the level curves exceed B.

Let $u = \gcd(d_1, \ldots, d_n) > 1$ be a factor of d_i and D.

We write, $d_i = uv_i$, $D = uv$, $v = \sum_{i=1}^{n} v_i$ and $r_i = \frac{v_i}{v}$, $I = 1, \ldots, n$.

We have, $E_m(i,(e-1)v_i+1)$

$= \left\lceil \frac{(e-1)v_i+1-\sqrt[m]{B}}{r_i} \right\rceil$, $e = 1, \ldots, u$

$= \left\lceil (e-1)v + \frac{1-\sqrt[m]{B}}{r_i} \right\rceil$

$= (e-1)v + \left\lceil \frac{1-\sqrt[m]{B}}{r_i} \right\rceil$

$> (e-1)v$ since $B < 1$.

Further, $L_m(i,ev_i)$
$= \left\lfloor \frac{ev_i - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor$
$= \left\lfloor ev + \frac{\sqrt[m]{B} - 1}{r_i} + 1 \right\rfloor$
$= ev + \left\lfloor \frac{\sqrt[m]{B} - 1}{r_i} + 1 \right\rfloor$
$\leq ev$.

Thus, $(e-1)v$
$< E_m(i, (e-1)v_i+1)$
$\leq L_m(i,(e-1)v_i+1)$
$\leq E_m(i, ev_i)$
$\leq L_m(i, ev_i)$
$\leq ev$.

This implies that v_i copies $(e-1)v_i + 1, \ldots, ev_i$ of model i, i = 1, …, n occupy positions in
$[(e-1)v+1, ev]$.

This shows that the sequence consists of u periods.
So, cyclic sequence is periodic. Each period is a subsequence of the sequence.
Now, we show that each period consists of the same models in the same order.
The e^{th} period of copies of model i, i = 1, …, n is labeled as $(e-1)v_i + f$, e = 1,…, u; f = 1, …, v_i .

For $E_m(i, j)$, $E_m(i, ev_i+f)$
$= \left\lceil \frac{ev_i + f - \sqrt[m]{B}}{r_i} \right\rceil$
$= \left\lceil \frac{(e-1)v_i + f - \sqrt[m]{B}}{r_i} + v \right\rceil$
$= \left\lceil \frac{(e-1)v_i + f - \sqrt[m]{B}}{r_i} \right\rceil + v$
$= E_m(i,(e-1)v_i + f) + v$
$= E_m(i, f) + ev$, for f = 1, …, v_i; I = 1, …, n.

For $L_m(i, j)$, $L_m(i,ev_i+ f)$
$= \left\lfloor \frac{ev_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor$
$= \left\lfloor \frac{(e-1)v_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 + v \right\rfloor$
$= \left\lfloor \frac{(e-1)v_i + f - 1 + \sqrt[m]{B}}{r_i} + 1 \right\rfloor + v$

$= L_m(i,(e-1)v_i+ f) + v$
$= L_m(i,f)+ ev$, for $f = 1, \ldots, v_i$; $i = 1, \ldots, n$.

The linear relations $E_m(i,ev_i+ f) = E_m(i,f) + ev$ and $L_m(i,ev_i+ f) = L_m(i,f) + ev$ imply that each period consists of v units of models and all units in thev isame order after sequencing the v units in the eth period.

An optimal subsequence can be determined for the first period. Then an optimal sequence consisting of u repetitions of this subsequence exists.

Corollary 1 The optimal bottlenecks of a subsequence and of its sequence are optimal.

Proof: Assume that a subsequence consists of D' copies with demands d_i' for model i, $i = 1, \ldots, n$
such that $\sum_{i=1}^{n} d_i'$ and $uD' = D$

We can write, $(x_{ik} - kr_i)^m$
$= (x_{i,(\theta D'+k')} - (\theta D' + k)r_i)^m$ for $0 \le \theta < u$ and $1 \le k' < D'$
$= (\theta d_i' + x_{ik'} - \theta d_i' - k'r_i)^m$
$= (x_{ik'} - k'r_i)^m$

4. Conclusion

The bottleneck product rate variation problem can be solved in pseudo-polynomial time. The complexity would be expensive for large size instances of the problem. So, it is natural to seek the cyclic sequence that is optimal. The cyclic sequence to the problem if exists is optimal.

The existence of optimal cyclic sequence for the bottleneck product rate variation problem with significant setup time and arbitrary processing time would be an area for future research.

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