The Nepali Math. Sc. Report
Vol. 29, No. 1 & 2, 2009

Parseval's Identity for Low-Dimensional Nilpotent Lie Groups $G_{5,6}$ and $G_{6,15}$

CHET RAJ BHATTA
Central Department of Mathematics
Tribhuvan University, Kirtipur
email: crbhatta@yahoo.com

Abstract. We prove the Parseval's identity for low-dimensional Nilpotent Lie groups such as $G_{5,6}$ and $G_{6,15}$ which are important for proving Hardy uncertainty principles type results.

Key Words: Fourier transform, Hilbert Schmidt norm, kernel function.

1. INTRODUCTION

Let g be an n-dimensional real Nilpotent Lie algebra and G = exp g be the associated connected and simply connected Nilpotent Lie group. Let $\{x_1, ..., x_n\}$ be a strong Malcev basis of g through the ascending central series of g. In particular, RX_1 is contained in the centre of g. We introduce a norm function on G by setting for

$$x = \exp(x_1X_1 + ... + x_nX_n) \in G,\ x_j \in \mathfrak{R}$$
$$||x|| = (x_1^2 + x_2^2 + ... + x_n^2)^{1/2}$$

The composed map

$$\mathfrak{R}^n \longrightarrow g \longrightarrow G,\ (x_1, ..., x_n) \longrightarrow \sum_{j=1}^{n} x_j X_j \longrightarrow \exp\left(\sum_{j=1}^{n} x_i X_j\right)$$

is a diffeomosphism and maps Lebesgue measure on $\Re^n$ to Haar measure on G. In this manner, we shall always identify g and sometimes G_1 as sets with $\Re^n$. The measurable (integrable) functions on G can be viewed as such functions on $\Re^n$. The measurable (integrable) functions on G can be viewed as such functional on $\Re^n$.

Let g denote the vector space dual of g and $\{X_1^*, ..., X_n^*\}$ the basis of g^* which is dual to $\{X_1, ..., X_n\}$. Then $\{X_1^*, ..., X_n^*\}$ is Jordon Holder basis for the coadjoint action of G on g^*. We shall identify g^* with $\Re^n$ via the map $\xi = (\xi_1, ..., \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$ and on g^*. We shall identify g^* with $\Re^n$ via the map $\xi = (\xi_1, ..., \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$ and on g^* we introduce the Eucledian norm relative to the basis $\{X_1^*, ..., X_n^*\}$, that is

$$\left\| \sum_{j=1}^{n} \xi_j X_j^* \right\| = (\xi_1^2 + \xi_2^2 + ... + \xi_n^2)^{1/2} = ||\xi||.$$

For an operator T in a Hilbert space such that T*T is a trace class. $||T||_{HS}$ will denote the Hillbert Schmidt norm of T.

2. THREAD LIKE NILPOTENT LIE GROUPS

For $n \geq 3$, let g_n be the n-dimensional real Nilpotent Lie algebra with basis $X_1, ..., X_n$ and non trivial lie brackets $[X_1, X_{n-1}] = X_{n-2}, ..., [X_1, X_2] = X_1$.

g_n is a $(n-1)$ step Nilpotent and is a product of RX_n and the abelian ideal $\sum_{j=1}^{n-1} RX_j$. Note that g_3 is the Heisenberg Lie algebra. Let $G_n = \exp(g_n)$.

For $\xi = \sum_{j=1}^{n-1} \xi_j X_j^* \in g_n^*$, the coadjoint action of G_n is given by

$$Ad^* (\exp(tX_n)) \xi = \sum_{j=1}^{n-1} P_j(\xi, t) X_j^*,$$

where for $i \leq j \leq n-1$, $P_j(\xi, t)$ is the polynomial in t defined by

$$P_j(\xi, t) = \sum_{k=1}^{j-1} (1/k!) (-1)^k t^k \xi_{j-k}$$

The orbit of ξ is generic with respect to the basis $\{X_1^*, ..., X_n^*\}$ if and only if $\xi_1 \neq 0$, and the jumping indices are 2 to n. The cross section X_{ξ_1} for the set of generic orbit is given by,

$$X_{\xi_1} = \{\xi = (\xi_1, 0, \xi_3, ..., \xi_{n-1}, 0) : \xi_1 \in \Re, \xi_1 \neq 0\}$$

For $\xi \in g_n^*$, let π_ξ denote the irreducible representation of G_n, associated with ξ. Then the mapping $\xi \to \pi_\xi$ is bijection of X_ξ and the set of all generic irredicible representation. Plancherel measure on $\hat{G}_n$ is supported by these π_ξ. Denoting by F the fourier transform on R^{n-1}, it follows that the Hilbert Schmidt norm of the operator. π_ξ (f) , $f \in L^1 \cap L^2 (G_n)$ is given by

$$\|\pi_\xi (f)\|^2_{HS} = \int_{R^2} F\, f\{p_1 (\xi, t), ..., P_{n-1} (\xi, t), t - s\}^2 \, ds\, dt$$

The following group of lower dimensions such as $G_{5,6}$ and $G_{6,15}$ are found in [8].

3. PARSEVAL IDENTITY FOR $G_{5,6}$

Let $G = G_{5,6} = R^5$

$$(x_1, ..., x_5)\, (y_1, ..., y_5)$$

$$= (x_1 + y_1 + x_4y_3 + x_5y_2 + x_4x_5y_4 + \frac{1}{2} x_5\, y_4^2 + \frac{1}{2} x_5^2\, y_3 + \frac{1}{6} x_2 + y_2 + x_5y_3 + \frac{1}{2} x_5^2\, y_4,$$
$$x_3 + y_3 + x_5y_4,\ x_4 + y_4,\ x_5)$$

$$(x_1, ..., x_5)^{-1} = (-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2} x_3x_5^2 - \frac{1}{2} x^2{}_4\, x_5 + \frac{1}{6} x_4x^3{}_5, -x_2 + x_3x_5$$
$$- \frac{1}{2} x_4\, x^2{}_5, -x_3 + x_4x_5, -x_4, -x_5)$$

For $y_1, y_2 \in R^2$

$$\pi_{\xi_1} (f)\, \phi\, (y_1, y_2) = \int_{\Re^5} f(x)\ \pi_{\xi_1}\ (-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}\, x_3x^2{}_5 - \frac{1}{2}\, x^2{}_4x_5 + \frac{1}{6}\, x_4x^3{}_5,$$
$$-x_2 + x_3x_5 - \frac{1}{2} x_4x^2{}_5, -x_3 + x_4x_5, -x_4, -x_5)\ \phi\ (y_1, y_2)\, dx$$
$$= \int_{\Re^5} f(x) \exp 2\pi i\, [-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2} x_3x^2{}_5 - \frac{1}{2} x^2{}_4\, x_5 + \frac{1}{6} x_4x^3{}_5$$
$$+ \frac{1}{2} x^2{}_4x_5 - \frac{1}{6} x_4x^3{}_5 - (-x_3 + x_4x_5)\, y_1 + x_4x_5y_1 - \frac{1}{6} x^3{}_5y_1 +$$

$$\frac{1}{2}x_5y^2{}_1 - (-x_2 + x_3x_5 - \frac{1}{2}x_4x^2{}_5)\,y_2 - \frac{1}{2}x_4x^2{}_5\,y_2 - \frac{1}{2}x_4x_5\,y^2{}_2 -$$

$$\frac{1}{2}x^3{}_5\,y^2{}_2 - \frac{1}{2}x^2{}_5y_1y_2 - \frac{1}{2}x_5y_1y^2{}_2]\ \phi\,(y_1 + x_4,\, y_2 + x_5)\,dx$$

$$x_4 \to x_4 - y_1,\ x_5 \to x_5 - y_2$$

$$= \int_{\Re^5} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\ \exp 2\pi i\,[-x_1 + x_2(x_5 - y_2)$$

$$+ x_3(x_4 - y_1) - \frac{1}{2}x_3\,(x_5 - y_2)^2 + x_3y_1 - \frac{1}{6}(x_5 - y_2)^3\,y_1 +$$

$$\frac{1}{2}(x_5 - y_2)\,y_1{}^2 + x_2y_2 - x_3(x_5 - y_2)\,y_2 - \frac{1}{2}(x_4 - y_1)\,(x_5 - y_2)\,y_2{}^2 -$$

$$\frac{1}{2}(x_5 - y_2)^3\,y_2{}^2 - \frac{1}{2}(x_5 - y_2)^2\,y_1y_2 - \frac{1}{2}(x_5 - y_2)\,y_1y_2{}^2)\xi_1]\ \phi(x_4, x_5)\,dx$$

$$= \int_{\Re^5} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\ \exp\,[2\pi i\,((-x_1 + x_2x_5 - x_3)\,[x_4 - \frac{1}{2}$$

$$(x_5 - y_2)^2 - (x_5 - y_2)\,y_2)\xi_1 + U(y_1, y_2, x_4, x_5)\,\xi_1)\ \phi(x_4, x_5)\,dx$$

$$K^f_{\xi_1}\,(y_1, y_2, x_4, x_5) = \int_{\Re^3} f(x_1, x_2, x_3, x_4 - y_1, x_5 - y_2)\ \exp - 2\pi i\,(x_1\xi_1 -$$

$$x_2x_5\xi_1 - x_3\,[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_1]\,\xi_1 -$$

$$U(y_1, y_2, x_4, x_5)\xi_1)\,dx_1\,dx_2\,dx_3$$

$$= F_{123}\,f(\xi_1, -x_5\,\xi_1, -[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_2]$$

$$\xi_1,\, x_4 - y_1,\, x_5 - y_2)\ \exp 2\pi\,i\,U(y_1, y_2, x_4, x_5)\,\xi_1$$

$$= \int_{\Re^4} |\,K^f_{\xi_1}\,(y_1, y_2, x_4, x_5)|^2\,dy_1\,dy_2\,dx_4\,dx_5$$

$$\|\pi_{\xi_1}(f)\|^2_{HS} = \int_{\Re^4} |F_{123}\,f(\xi_1, -x_5\xi_1, -[x_4 - \frac{1}{2}(x_5 - y_2)^2 - (x_5 - y_2)\,y_2]$$

$$\xi_1,\, x_4 - y_1,\, x_5 - y_2]\,dy_1\,dy_2\,dx_4\,dx_5$$

$$x_5 \to \frac{-1}{\xi_1}x_5,\ x_4 \to \frac{-1}{\xi_1}x_4$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{123}\, f(\xi_1, x_5, x_4 + \left(\frac{1}{2}\left(-\frac{x_5}{\xi_1} - y_2\right)^2 + \left(\frac{-x_5}{\xi_1} - y_2\right) y_2\right) \xi_1$$

$$-\left(\frac{1}{\xi_1} x_4 - y_1 - \frac{x_5}{\xi_1} - y_2\right) dy_1\, dy_2\, dx_4\, dx_5$$

$$y_1 \to -y_1 - \frac{1}{\xi_1} x_4,\ y_2 \to -y_2 - \frac{1}{\xi_1} x_5$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{123}\, f(\xi_1, x_5, x_4 + \left(\frac{1}{2} y_2{}^2 - y_2 \left(y_2 + \frac{1}{\xi_1} x_5\right)\right)$$

$$\xi_1, y_1, y_2)|^2\, dy_1\, dy_2\, dx_4\, dx_5$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{13}\, f(\xi_1, u_1\, x_4 + \left(\frac{-1}{2} y_2{}^2\, \xi_1 - y_2\, u\right) y_1, y_2)|^2$$

$$dy_1 dy_2\, dx_4\, du$$

$$x_4 \to x_4 + \frac{1}{2} y_2{}^2\, \xi_1 + y_2\, u$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_{13}\, f(\xi_1, u_1\, x_4, y_1, y_2)|^2\, dy_1\, dy_2\, dx_4\, du$$

$$= \frac{1}{\xi_1{}^2} \int_{\Re^4} |F_1\, f(\xi_1\, u, w, y_1, y_2)|^2\, dy_1\, dy_2\, dw\, du$$

4. PARSEVAL IDENTITY FOR $G_{6,15}$

$G = G_{6,15} = \Re^6$

$(x_1, ..., x_6)\,(y_1, ..., y_6) = (x_1 + y_1 + x_6 y_4,\ x_2 + y_2 + x_5\, y_4,\ x_3 + y_3 + x_6 y_5,\ x_4 + y_4,\ x_5 + y_5,\ x_6 + y_6)$

$(x_1, x_2, ..., x_6)^{-1} = (-x_1 + x_4 x_6, -x_2 + x_4 x_5, -x_3 + x_5 x_6, -x_4, -x_5, -x_6)$

For $\phi \in L^2(\Re)$

$\hat{f}\,(\pi_{\xi_1, \xi_2, \xi_3, \xi_6})\, \phi\,(y),\ \xi_2 \neq 0$

$$= \int_{\Re^6} f(x)\, \pi_{\xi_1, \xi_2, \xi_3, \xi_6}\, (-x_1 + x_4 x_6, -x_2 + x_4 x_5, -x_3 + x_5 x_6, -x_4, -x_5, -x_6)\, \phi\,(y)\, dx$$

$$= \int_{\Re^6} f(x) \exp 2\pi i\, [(-x_1 + x_4 x_6)\, \xi_1 + (-x_2 + x_4 x_5)\, \xi_2 + (-x_3 + x_5 x_6 - x_5 x_6)\, \xi_3 + \frac{\xi_3}{\xi_2}$$

$$(-x_6 y - \frac{1}{2} x^2{}_6\, \xi_1) - x_6\xi_1 + x_4 y]\, \phi(y + \xi_2 x_5 + \xi_1 x_6)\, dx$$

Applying $x_5 \to \frac{1}{\xi_2}(x_5 - y - \xi_1 x_6)$

$$= \frac{1}{|\xi_2|} \int_{\Re^6} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5, -y - \xi, x_6), x_6) \exp [(-x_1 + x_4 x_6)\, \xi_1 + (-x_2 + x_4$$

$$(\frac{1}{\xi_2}(x_5 - y - \xi_1 x_6))\, \xi_2 + (- x_3\, \xi_3) + \frac{\xi_3}{\xi_2}(- x_6 y - \frac{1}{2} x^2{}_2\, \xi_1) - x_6\xi_6 + x_4 y]\, \phi\, (x_5)\, dx$$

$$= \frac{1}{|\xi_2|} \int_{\Re^6} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5 - y - \xi_1, x_6), x_6) \exp.\, 2\pi i\, [- x_1\xi_1 - x_2\xi_2 - x_3\xi_3 +$$

$$x_4 x_5 + \frac{\xi_3}{\xi_2}(- x_6 y - \frac{1}{2} x_6{}^2\, \xi_1) - x_6\xi_6]\, \phi\, (x_5)\, dx$$

$\hat{f}\,(\pi_{\xi_1, \xi_2, \xi_3, \xi_6})$ is the integral operator on $L^2\,(\Re)$ where kernel is

$$K^f{}_{(\xi_1, \xi_2, \xi_3, \xi_6)}\,(y, x_5) = \frac{1}{|\xi_2|} \int_{\Re^5} f(x_1, x_2, x_3, x_4, \frac{1}{\xi_2}(x_5 - y - \xi_1 x_6), x_6) \exp.\,(2\pi i)$$

$$[\sum_{i=1}^{3} x_i\xi_i - x_4 x_5 + \frac{\xi_3}{\xi_2}[x_6 y + \frac{1}{2} \int_{\Re^5} x^2{}_6\, \xi_1 + x_6\xi_6]\, dx_1\, dx_2\, dx_3\, dx_4\, dx_6$$

$$= \frac{1}{|\xi_2|} \int_{\Re} F_1F_2F_3F_4\, (\xi_1, \xi_2, \xi_3, x_5\, \frac{1}{\xi_2}(x_5 - y - \xi_1 x_6), x_6) \exp\,(- 2\pi i)$$

$$[\frac{\xi_3}{\xi_2}(x_6 y + \frac{1}{2} x^2{}_6\, \xi_1) + x_6 x_6]\, dx_6$$

$$\|\hat{f}\,(\pi_{\xi_1, \xi_2, \xi_3, \xi_6})\|^2 = \int_{\Re^2} \|k^f{}_{(\xi_1, \xi_2, \xi_3, \xi_6)}\,(y, x_5)|^2\, dy\, dx_5$$

$$= \frac{1}{|\xi_2|} \int_{\Re^2} \Big| \int_{\Re} F_1\, F_2\, F_3\, F_4\, (\xi_1, \xi_2, \xi_3, -x_5, \frac{1}{\xi_2}(x_5 - y - \xi_1\, x_6, x_6) \exp.\,(2\pi i)$$

$$(\frac{\xi_3}{\xi_2}(x_6 y + \frac{1}{2} x^2{}_6\;\; \xi_1) + x_3\xi_6)\, dx_6|^2\, dy\, dx_5$$

$$y \to y - \frac{1}{2} x_6\, \xi_1$$

$$= \frac{1}{|\xi_2|} \int_{\Re^2} \left| \int_{\Re} F_1F_2F_3F_4 \left(\xi_1, \xi_2, \xi_3, -x_5, \frac{1}{\xi_2}(x_5 - y - \frac{1}{2}\xi_1 x_6), x_6\right) \exp(-2\pi i) \right.$$

$$\left. \left(\frac{\xi_3}{\xi_2}(x_6 y + x_6\xi_6)\right) dx_6 \right|^2 dy\, dx_5$$

REFERENCES

1. Bhatta, C.R., Parsevals Identity for low-dimensional Nilpotent Liegroups $G_{5,1}$, $G_{5,3}$, $G_{5,3}$ and $G_{5,5}$ Kathmandu University Journal of Science, Engineering and technology, vol. 6 No. 1, January, PP 22-23, (2009).
2. Folland, G., B., Sitaram, A. The Uncertainty Principle: A Mathematical Survey, J. Fourier Analysis and Application, 3, 207 – 308 (1997).
3. Folland, G.B., A course in Harmonic Analysis, CRC Press, (1995).
4. H. Reiter & J. D. Stegeman, Classical Harmonic Analysis and locally compact groups, Oxford University Press (2000).
5. Hewitt, E., and Ross, K.A., Abstract Harmonic Analysis I & II. Springer Verlag, (1963 & 1970).
6. Kaniuth, E. A. Hardy theorem for simply connected nilpotent ciegroups, preprint, (2000).
7. Kumar, A., and Bhatta, C.R. An Uncertainty Principle Like Hardy's Theorem for Nilpotent Liegroups, J. Aust. Math Soc, 76, 1-7, (2004).
8. L. Corwin, L. and Green leaf, F.P. Representation of Nilpotent Liegroups and their applications, Part I, Basic Theory and Examples (Cambridge University Press), (1990).
9. Ole, A. elson. Unitary representations and coadjoint orbits of low dimensional Nilpotent Lie groups, Queens papers in pure and applied mathematics, No. 63, (1983).