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Related fixed point theorems for set-valued mappings on two complete and compact metric spaces

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Abstract: A related fixed point theorem for set-valued mappings on two complete metric spaces is obtained. A generalization for two compact metric spaces is also obtained.

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1. Introduction

The following theorem was proved by Namdeo, Tiwari, Fisher and Tas [3].

Theorem 1.1: *Let (X,d) and (Y,ρ) be complete metric spaces, let T be mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$d(Sy, Sy')\, d(STx, STx') \leq c \max\{d(Sy,Sy')\, \rho(Tx,Tx'),$$
$$d(x',Sy)\, \rho(y',Tx),\ d(x,x')\, d(Sy,Sy'),\ d(Sy,STx)\, d(Sy',STx')\}$$

$$\rho(Tx,Tx')\, \rho(TSy,TSy') \leq c \max\{d(Sy,Sy')\, \rho(Tx,Tx'),$$
$$d(x',Sy)\, \rho(y',Tx),\ \rho(y,y')\, \rho(Tx,Tx'),\ \rho(Tx,TSy)\, \rho(Tx',TSy')\}$$

for all x, x' in X and y, y' in Y where $0 \leq c < 1$. If either T or S is continuous then ST has a unique fixed point z in X and TS has a unique fixed point v in Y. Further $Tz = w$ and $Sw = z$.

The following theorem was proved by Fisher and Turkoglu [4].

Theorem 1.2: *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let S be mapping of X into $B(Y)$ and R be mapping of Y into $B(X)$ satisfying the following inequalities,*

$$\delta_1(RSx, RSx') \leq c \max\{d_1(x,x'), \delta_1(x, RSx), \delta_1(x', RSx'), \delta_2(Sx, Sx)\}$$

$$\delta_2(RSy, RSy') \leq c \max\{d_2(y,y'), \delta_2(y, SRy), \delta_2(y', SRy'), \delta_1(Ry, Ry')\}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If S is continuous then RS has a unique fixed point u in X and SR has a unique fixed point v in Y.

The aim of our work is to generalize theorem 1.1 by considering two set-valued mappings and two complete metric spaces.

Before comming to our main result we recall the following from Fisher [1, 2].

(i) The function $\delta(A, B)$ with A and B in $B(X)$ is defined by
$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$.

(ii) If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$

(iii) If B also consists of a single point b we write $\delta(A, B) = \delta(a, b) = d(a, b)$
If follows easily from the definition that $\delta(A, B) = \delta(B, A) \geq 0$,
$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all A, B and C in $B(X)$.

Now let $\{A_n : n = 1, 2, 3, \ldots\}$ be sequence of non-empty subsets of X. We say that the sequence $\{A_n\}$ converges to the subset A of X if

(iv) Each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n = 1, 2, 3, \ldots$.

(v) For arbitrary $\varepsilon > 0$, there exists an integer N such that $A_n \subset A_\varepsilon$ for $n > N$, where A_ε is the union of all open spheres with centres in A and radius ε. A is then said to be the limit of the sequence $\{A_n\}$.

The following lemma was proved in Fisher [1].

Lemma 1.3: *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets of A and B respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Now, let F be a mapping of X into $B(X)$. We say that the mapping F is continuous at a point in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$. We say that F is continuous mapping of X into $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a fixed point of F if z is in Fz. If a is in $B(X)$, we define the set $FA = U_{a \in A}\ Fa$.

Main Results:

We prove the following theorems.

Theorem 2.1: *Let (X, d_1) and (Y, d_2) be two complete metric spaces, let T be a mapping of X into $B(Y)$ and let S be a mapping of Y into $B(X)$ satisfying inequalities,*

$$\delta_1(Sy,Sy')\ \delta_1(STx,STx') \le c \max\{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx),$$

$$d_1(x,x')\ \delta_1(Sy,Sy'),\ \delta_1(Sy',STx)\delta_1(Sy',STx)\} \tag{1}$$

$$\delta_2(Tx,Tx')\ \delta_2(TSy,TSy') \le c \max\{\delta_1(Sy,Sy')\ \delta_2(Tx,Tx'),\ \delta_1(x',Sy)\ \delta_2(y',Tx),$$

$$d_2(y,y')\ \delta_2(Tx,Tx'),\ \delta_2(Tx,TSy)\ \delta_2(Tx',TSy')\} \tag{2}$$

for all x, x' in X and y, y' in Y where $0 \le c < 1$. If T is continuous, then ST has a unique fixed point x in X and TS has a unique fixed point w in Y.

Proof: *Let x_1 be an arbitrary point in X. Define sequence $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows. Choose a point y_1 in Tx_1 and a point x_2 in Sy_1. In general, having choosen x_n in X and y_n in Y, choose x_{n+1} in Sy_n and then y_{n+1} in Tx_{n+1} for* $n = 1, 2, 3, \ldots$.

Then,

$$\begin{aligned} d_1(x_{n-1}, x_n)\ d_1(x_n,x_{n+1}) &= \delta_1(Sy_{n-2},Sy_{n-1})\ \delta_2(STx_{n-1},STx_n) \\ &\le c \max\{d_1(x_{n-1},x_n)\ d_2(y_{n-1},y_n),\ d_1(x_n,x_{n-1})\ d_2(y_{n-1},y_{n-1}), \\ &\qquad d_1(x_{n-1},x_n)\ d_1(x_{n-1},x_n),\ d_1(x_{n-1},x_n)\ d_2(x_n,x_{n+1})\}. \end{aligned}$$

From which is follows that

$$d_1(x_n, x_{n+1}) \le c \max\{d_2(y_{n-1},y_n), d_1(x_{n-1},x_n)\} \tag{3}$$

Applying inequality (2), we get

$$\begin{aligned} [d_2(y_{n-1},y_n)]^2 &= \delta_2(Tx_{n-1},Tx_n),\ \delta_2(TSy_{n-2},TSy_{n-1}) \\ &\le c \max\{d_2(x_{n-1},x_n)\ d_2(y_{n-1},y_n),\ d_1(x_n,x_{n-1})\ d_2(y_{n-1},y_{n-1}),\ d_2(y_{n-2},y_{n-1}) \\ &\qquad d_2(y_{n-1},y_n),\ d_2(y_{n-1},y_{n-1})\ d_2(y_{n-1},y_n)\} \end{aligned} \tag{4}$$

from which is follows that $d_2(y_{n-1},y_n) \le c \max\{d_1(x_{n-1},x_n),\ d_2(y_{n-2},y_{n-1})\}$.

It now follows easily by induction that

$$d_1(x_n, x_{n+1}) \le c^n \max\{d_1(x, x_1),\ d_2(y_1, y_2)\}$$

$$d_2(y_{n-1}, y_n) \le c^{n-2} \max\{d_1(x_1, x_2),\ d_2(y, y_1)\}$$

for $n = 1, 2, \ldots$. Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X with w in Y.

Applying inequality (1), we have

$$\delta_1(Sw, x_{n+1})\ \delta_1(STz, x_{n+1}) = \delta_1(Sw, Sy_n)\ \delta_1(STz, STx_n)$$

$$\le c \max\{\delta_1(Sw, x_n)\ \delta_2(Tz, y_n),\ \delta_1(x_n, Sw)\ \delta_2(y_n, Tz),\ d_1(z, x_n)\ \delta_1(Sw, x_n),$$

$$\delta_1(Sw, STz)\ \delta_1(x_{n+1}, x_{n+1})\}$$

Letting n tends to infinity we have

$$\delta_1(Sw, z)\delta_1(STz, z) \le c \max(\delta_1(Sw, z)\ \delta_2(Tz, w),$$

$$\delta_1(z, Sw)\ \delta_2(w, Tz),\ d_1(z, z)\ \delta_1(Sw, z)\}$$

From which it follows that

(5) $$\delta_1(Sw, z)\ \delta_1(STz, z) \le c\{\delta_1(Sw, z)\ \delta_2(Tz, w)\}$$

and so either $Sw = z$ or

(6) $$\delta_1(STz, z) \le c\ \delta_2(Tz, w)$$

Applying inequality (2), we have

$$\delta_2(Tz, y_n)\ \delta_2(TSw, y_{n+1}) = \delta_2(Tz, Tx_n)\ \delta_2(TSw, TSy_n)$$

$$\le c \max\{\delta_1(Sw, x_{n+1})\ \delta_2(Tz, y_n),\ \delta_1(z, Sw)\ \delta_2(w, Tz),$$

$$d_2(w, y_n)\ \delta_2(Tz, y_n),\ \delta_2(Tz, TSw)\ \delta_2(y_n, TSw)\}$$

Letting n tends to infinity we have

(7) $$\delta_2(Tz, w)\ \delta_2(TSw, w) \le c\delta_1(Sw, z)\ \delta_2(Tz, w)$$

and so either $Tz = w$

Theorem 2.3: *Let (X, d) and (Y, ρ) be compact metric spaces, let T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities*

$$d(Sy, Sy')\ d(STx, STx') < \max\{d(Sy, Sy')\ \rho(Tx, Tx'),\ d(x', Sy)\ \rho(y', Tx),$$

$$d(x, x')\ d(Sy, Sy'), d(Sy, STx)\ d(Sy', STx')\}$$

$$\rho(Tx, Tx')\ \rho(TSy, TSy') < \max\{d(Sy, Sy')\ \rho(Tx, Tx'), d(x', Sy)\ \rho(y', Tx),$$

$$\rho(y, y')\ \rho(Tx, Tx'),\ \rho(Tx, TSy)\ \rho(Tx', TSy')\}$$

for all x, x' in Y and y, y' in Y. Then ST has a unique fixed point z in X and TS has a **unique fixed point w in Y. Further,** $Tz = w$ and $Sw = z$.

We now prove the following fixed point theorem for compact metric spaces.

Theorem 2.4: *Let (X, d_1) and (Y, d_2) be two compact metric spaces, let T be a continuous mapping of X into $B(Y)$ and let S be a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta_1(Sy, Sy')\,\delta_1(STx, STx') < \max\{\delta_1(Sy,Sy')\,\delta_2(Tx,Tx'),\ \delta_1(x',Sy)\,\delta_2(y',Tx),$$
$$d_1(x,x')\,\delta_1(Sy,Sy'),\ \delta_1(Sy,STx)\,\delta_1(Sy',STx')\} \tag{11}$$

$$\delta_2(Tx, Tx')\,\delta_2(TSy, TSy') < \max\{\delta_1(Sy,Sy')\,\delta_2(Tx,Tx'),\ \delta_1(x',Sy)\,\delta_2(y',Tx),$$
$$d_2(y,y')\,\delta_2(Tx,Tx'),\ \delta_2(Tx,TSy)\,\delta_2(Tx',TSy')\} \tag{12}$$

for all x, x' in X and y, y' in Y. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Proof: Firstly, let us assume that there is no $a < 1$ such that

$$\delta_1(Sy,STSy)\,\delta_1(STx,STSTx) \le a \max\{\delta_1(Sy,STSy)\,\rho(Tx,TSTx),$$
$$\delta_1(STx, Sy)\,\delta_2(TSy,Tx),\ \delta_1(x,STx)\,\delta_1(Sy,STSy),$$
$$\delta_1(Sy,STx)\,\delta_1(STSy,STSTx)\} \tag{13}$$

for all x in X and y in Y. Then there exist sequences $\{x_n\}$ in X and $\{y_n\}$ in Y such that

$$\delta_1(Sy_n, STSy_n)\,d(STx_n, STSTx_n) > (1 - n^{-1}) \max\{\delta_1(Sy_n, STSy_n)\,\delta_2(Tx_n, TSTx_n),$$
$$\delta_1(STx_n,Sy_n)\,\delta_2(TSy_n,Tx_n),\ \delta_1(x_n,STx_n)\,\delta_1(Sy_n,STSy_n),$$
$$\delta_1(Sy_n,STx_n)\,\delta_1(STSy_n,STSTx_n)\} \tag{14}$$

for $n = 1, 2, \ldots$. Since X and Y are compact, and by relabelling if necessary, we may suppose that the sequence $\{x_n\}$ converges to z' in X and the sequence $\{y_n\}$ converges to w' in Y. Letting n tends to infinity in inequality (14), it follows that

$$\delta_1(Sw',STSw')\,\delta_1(STz',STSTz') \ge \max\{\delta_1(Sw',STSw')\,\delta_2(Tz',TSTz'),$$
$$\delta_1(STz',Sw')\,\delta_2(TSw',Tz'),\ \delta_1(z',STz')\,\delta_1(Sw',STSw'),$$
$$\delta_1(Sw',STz')\,\delta_1(STSw',STSTz')\} \tag{15}$$

This is only possible if the right hand side of inequalities (15) is zero. It follows that either

$$STz' = STSTz' \text{ or } Sw' = STSw'.$$

If $STz' = STSTz'$, then $STz' = z$ is a fixed point of ST and it follows that $Tz = w$ is a fixed point of TS.

If $Sw' = STWw'$, then $Sw' = z$ is a fixed point of ST and it again follows that $Tz = w$ is a fixed point of TS.

Secondly, let us assume that there exists no $b < 1$ such that

$$\delta_2(Tx, TSTx)\ \delta_2(TSy, TSTy) \leq b \max \{\delta_1(Sy, STSy)\ \delta_2(Tx, TSTx),$$
$$\delta_1(STx, Sy)\ \delta_2(TSy, Tx),\ \delta_2(y, TSy)\ \delta_2(Tx, TSTx),$$
$$\delta_2(Tx, TSy)\ \delta_2(TSTx, STSTy)\} \tag{16}$$

for all x in X and y in Y. Then it follows that ST has a fixed point z and TS has a fixed point w.

Finally, suppose that there exist $a, b < 1$ satisfying (13) and (16). Then with $c = \max \{a, b\}$, it follows that if the sequences $\{x_n\}$ and $\{y_n\}$ are defined as in the proof of Theorem 2.1, inequalities (3) and (4) will hold. It then follows as in the proof of theorem 2.1 that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y. Since ST and TS are continuous, it now follows that z is a fixed point of ST and w is a fixed point of TS.

To prove the uniqueness, suppose that ST has a second distinct common fixed point z'. Then applying inequality (11) we have

$$[d_1(z, z')]^2 = [\delta_1(STz, STz')]^2 < \max \{d_1(z, z')\ d_2(Tz, Tz'),\ [d_1(z, z')]^2\}$$

which implies that

$$d_1(z, z') < \delta_2(Tz, Tz') \tag{17}$$

Further, applying inequalities (12) we have

$$[\delta_2(Tz, Tz')]^2 = \delta_2(Tz, Tz')\ \delta_2(TSTz, TSTz') < \max \{d_1(z, z')\ \delta_2(Tz, Tz'),$$
$$[\delta_2(Tz, Tz')]^2\}$$

which implies that

$$\delta_2(Tz, Tz') < d_1(z, z') \tag{18}$$

It now follows from inequalities (17) and (18) that

$$d_1(z, z') < \delta_2(Tz', Tz) < d_1(z, z')$$

which is a contradiction and hence the fixed point z must be unique.

Similarly, we can prove the uniqueness of w. This completes the proof of the theorem.

Corollary 2.5: *Let (X,d) be a compact metric space and let T be a continuous mapping of X into $B(X)$ satisfying the inequality*

$$\delta(Ty,Ty')\,\delta(T^2x,T^2x') < \max\{\delta(Ty,Ty')\,\delta(Tx,Tx'),\ \delta(x',Ty)\,\delta(y',Tx),$$
$$d(x,x')\,\delta(Ty,Ty'),\ \delta(Ty,T^2x)\,\delta(Ty',T^2x')\}$$

for all x, x', y, y' in X for which the right hand side of the inequality is positive. Then T has a unique fixed point z in X.

REFERENCES

[1] Fisher, B., Set-valued mappings on metric spaces. Fund. Math 112(1981), 141–145.

[2] Fisher, B., *Common fixed points of mappings and set-valued mappings. Rostock Math. Kolloq.* 18(1981) 69–77.

[3] Tiwari, Namdeo., Fisher and Tas, *Related fixed point theorems on two complete and compact metric spaces.* Internat. *J. Math.* and *Math. Sci*, Vol. 21, No. 3 (1998), 559–564.

[4] Fisher, B., and Turkoglu, D., Related fixed points for set valued mappings on two metric spaces. Internat. *J. Math* and *Math. Sci.* Vol. 23, No. 3(2000), 205–210.