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Remarks on fixed point theorem under implicit relations

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Abstract: In this paper, a common fixed point theorem satisfying an implicit relation, is established by removing the reciprocal continuity, relaxing the compatibility partly and replacing the completeness of the space with a set of four alternative natural conditions. Some related results and illustrative examples are also discussed.

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1. Introduction

Sessa [15] initiated the tradition of improving commutativity conditions in fixed point theorems by introducing the notion of 'weakly commuting mappings' which asserts that a pair of self-mappings (S, I) of a metric space (X, d) is said to be ***weakly commuting*** if, $d(SIx, ISx) \le d(Ix, Sx)$ for all x in X. It is noted that every commuting pair is weakly commuting but not conversely as shown in Sessa [15]. Jungck [6] also enlarged this class of weakly commuting mappings by defining 'compatible mappings' which asserts that a pair of self-mappings (S,I) is said to be ***compatible*** if, $\lim_{n\to\infty} d(SIx_n, ISx_n) = 0$ whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = t \in X$.

Recently, Jungck and Rhoades [7] (also Dhage [2]) termed a pair of self-mappings to be coincidentally commuting (or weakly compatible) if they merely commute at their coincidence points. One may note that this definition never needs to involve metric of the underlying set. The following one-way implication is obviously true but not conversely.

Commuting maps $\Rightarrow$ Weakly commuting maps $\Rightarrow$ Compatible Maps $\Rightarrow$ Coincidentally commuting maps.

Very recently Popa [11, 12, 13, 14] proved interesting fixed point theorems satisfying suitable implicit relations. For proving such results, Popa considers Φ to be the set of all continuous functions $F: (R^+)^6 \to R$ satisfying the following conditions:

F_1: F is non-increasing in t_5 and t_6.

F_2: there exists $h \in (0, 1)$ such that for $u,v \geq 0$ with

$F_{2(a)}$: $F(u, v, v, u, u+v, 0) \leq 0$ or

$F_{2(b)}$: $F(u, v, u, v, 0, u+v) \leq 0$

implies that $u \leq hv$.

The following examples of such functions F satisfying F_1 and F_2 appear in Popa [11, 12].

Example 1.1.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1(t_3^2 + t_4^2)/(t_3 + t_4) - a_2 t_2 - a_3(t_5 + t_6)$ where $a_i \geq 0$ ($i = 1, 2, 3$) with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

Example 1.2.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (c\, t_3 t_4 + b\, t_5 t_6)/(t_3 + t_4) - a\, t_2$ where $a, b, c \geq 0$ and $1 < c + 2a < 2$.

Example 1.3.[12] $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 + t_2^2 + t_1 - a(t_3^3 + t_4^3 + t_2 t_5 t_6)/(t_3^2 + t_4^2)$ where $a \in (0, 1)$.

Here, we give some natural examples of implicit condition & functions satisfying the conditions F_1 and F_2, which further strengthen the significance of employing implicit functions as it improves a class of contractive conditions.

Example 1.4. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - (a_1 t_3 t_4)/(t_3 + t_4) - a_2 t_2 - a_3(t_5 + t_6)$ with a_i positive and with at least one a_i ($i = 1, 2, 3$) is not zero satisfying $a_1 + 2a_2 + 4a_3 < 2$.

Then,

F_1: Obvious

F_2: For $v > 0$ and $F(u, v, v, u, u+v, 0) = u - (a_1 u v)/(u+v) - a_2 v - a_3(u+v) \leq 0$, then $u(u+v) - a_1 uv - a_2 v(u+v) - a_3(u+v)^2 \leq 0$.

If we set $f(t) = (1 - a_3)\, t^2 + (1 - a_1 - a_2 - 2a_3)\, t - a_2 - a_3$, where $t = u/v$. Then, since $f(0) = -a_2 - a_3 < 0$ and $f(1) = 2 - a_1 - 2a_3 - 4a_3 > 0$, so there exists a positive root 'h' of the equation $f(t) = 0$ with $h \in (0, 1)$. Then $f(t) \leq 0$ for $0 < t \leq h$. Thus, we have $u \leq hv$ which establishes $F_{2(a)}$. In case $u = 0$, we have $u \leq hv$. Similarly, one can also establishes $F_{2(b)}$.

Example 1.5. $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - a_1(t_5 + t_6)t_2/(t_3 + t_4) - a_2(t_2 + t_4)$ with a_i positive and with at least one a_i ($i = 1, 2, 3$) is non zero satisfying $a_1 + 2a_2 < 1$.

Then,

F_1: Obvious

F_2: For $v > 0$ and $F(u, v, v, u, u+v, 0) = u - a_1 v - a_2(u+v) \leq 0$, then $(1 - a_2)u - (a_1 + a_2)v \leq 0$.

If we set $f(t) = (1 - a_2)\, t - (a_1 + a_2)$, where $t = u/v$. Then, since $f(0) = a_1 - a_2 < 0$ and $f(1) = 1 - a_1 - 2a_2 > 0$, so as earlier, it can be shown that $u \leq hv$ with $h \in (0, 1)$ which establishes $F_{2(a)}$. Also, if $u = 0$, then $u \leq h\, v$. Similarly, one can also establishes $F_{2(b)}$.

The following fixed point theorems are proved in [12, 13]

Theorem 1.1. [5] *Let S, T, I, J be self mappings of a complete metric space (X,d) satisfying $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for each x, y in X, either*

$$d(Sx, Ty) \leq \alpha\, [\{d(Sx, Ix)\}^2 + \{d(Ty, Jy)\}^2] / [\, d(Sx, Ix) + d(Ty, Jy)] + \beta\, d(Ix, Jy)$$

if, $d(Sx, Ix) + d(Ty, Jy) \neq 0$, $\alpha, \beta > 0$ and $\alpha + \beta < 1$ or

$d(Sx, Ty) = 0$ if, $d(Sx, Ix) + d(Ty, Jy) = 0$.

If either (*a*) (S, I) are compatible, S or I is continuous and (T, J) are weakly compatible or (*b*) (T, J) are compatible, T or J is continuous and (S,I) are weakly compatible, then S, T, I and J have unique fixed point.

Recently, in an attempt to improve Theorem 1.1, Popa proved the following fixed point theorem via implicit relations.

Theorem 1.2.[13] *Let (S, I) and (T, J) be a weakly compatible pair of self-mappings of a complete metric space (X, d) satisfying $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for each x, y in X,*

$$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0,$$

with $d(Sx, Ix) + d(Ty, Jy) \neq 0$, where $F \in \Phi$, or

$$d(Sx, Ty) = 0 \text{ if, } d(Sx, Ix) + d(Ty, Jy) = 0.$$

Then, If (S, I) or (T, J) is a compatible pairs of reciprocally continuous mappings, then S, T, I and J have unique fixed point.

The main purpose of this paper is to improve Theorem 1.2 besides discussing related results and illustrative examples to demonstrate the utility of the results as remarks. Pant [10] introduced the concept of reciprocal continuity and it is important to note that continuity implies reciprocal continuity but not conversely. So, in this theorem, we relax the reciprocal continuity and compatibility conditions of the maps completely, weaken the completeness condition of the space to four alternative natural conditions and also deduce some important corollaries.

2. Main Results

Theorem 2.1. *Let (S, I) and (T, J) be a weakly compatible pair of self-mappings of a metric space (X, d) such that*

(i) *$S(X) \subset J(X)$ and $T(X) \subset I(X)$ and*

(ii) *$F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \leq 0$,*
for each x, y in X, with $d(Ix, Sx) + d(Jy, Ty) \neq 0$, where $F \in \Phi$, or
$d(Sx, Ty) = 0$ if, $d(Ix, Sx) + d(Jy, Ty) = 0$.

If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete subspace of X, then S, T, I and J have unique fixed point.

Proof. *Let x_0 be an arbitrary point in X, then since* (i) *holds, so we can inductively define sequences $\{x_n\}$ and $\{y_n\}$ by*

(1) $$y_{2n} = Sx_{2n} = Jx_{2n+1}\,;\, y_{2n+1} = Tx_{2n+1} = Ix_{2n+2} \text{ for } n = 0, 1, 2, \ldots$$

If $d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1}) \neq 0$ for $n = 0, 1, 2, \ldots$, then using inequality (ii), we have successfully

$$F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}),$$
$$d(Jx_{2n+1}, Sx_{2n})) \leq 0.$$

That is, $F(d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}),$
$$d(Tx_{2n+1}, Sx_{2n+1}) + d(Sx_{2n+1}, Tx_{2n+1}), 0) \le 0.$$

By (F_a), we have $d(Sx_{2n}, Tx_{2n+1}) \le h\, d(Sx_{2n}, Tx_{2n+1})$.

Similarly, if $d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1}) \ne 0$, for $n = 0, 1, 2, \ldots$, then by (F_b), we have $d(Sx_{2n}, Tx_{2n-1}) \le hd(Sx_{2n-2}, Tx_{2n-1})$ and so $d(Sx_{2n}, Tx_{2n+1}) \le h^{2n} d(Sx_0, Tx_1)$.

By a routine calculation, it follows that $\{y_n\}$ is a Cauchy sequence.

Now, suppose $J(X)$ is a complete subspace of X, then the subsequence $Jx_{2n+1} = Sx_{2n}$ is contained in $J(X)$ and hence there exists a limit u. Let $v \in J^{-1}u$, then $Jv = u$. Also, the subsequence $Ix_{2n+2} = Tx_{2n+1}$ converges to u. We prove that $Tv = u$.

Suppose on the contrary that $d(u, Tv) > 0$. Then setting $x = x_{2n}$ and $y = v$ in (ii), we get

$$F(d(Sx_{2n}, Tv), d(Ix_{2n}, Jv), d(Ix_{2n}, Sx_{2n}), d(Jv, Tv), d(Ix_{2n}, Tv), d(Jv, Sx_{2n})) \le 0,$$

which, on letting $n \to \infty$, reduces to $F(d(u, Tv), 0, 0, d(u, Tv), d(u, Tv), 0) \le 0$. This implies that $d(u, Tv) \le 0$. Thus, we have $u = Tv$. Hence, J and T have a point of coincidence. Since $T(X) \subset I(X)$, $Tv = u$ implies that $u \in I(X)$. Let $w \in I^{-1}u$, then $Iw = u$. Now, using the same argument, we can prove that $Sw = u$. Thus, S and I have a point of coincidence. If one assumes that $I(X)$ is complete, then analogous arguments establish the earlier conclusions.

If $S(X)$ is complete then by (i), we have $u \in S(X) \subset J(X)$. Similarly, if $T(X)$ is complete, then $u \in T(X) \subset I(X)$. Since the pairs (S, I) and (T, J) are weakly compatible and so coincidentally commuting at w and u and therefore, we have

$$u = Tv = Jv = Sw = Iw\,;\ Su = SIw = ISw = Iu \text{ and } Tu = TJv = JTv = Ju.$$

If $Tu \ne u$, then $d(Tu, u) > 0$ and hence,

$F(d(Sw, Tu), d(Iw, Ju), d(Iw, Sw), d(Ju, Tu), d(Iw, Tu), d(Ju, Sw)) = F(d(u, Tu), d(u, Tu), 0, 0, d(u, Tu), d(Tu, u)) > 0$, which is a contradiction (ii) and hence $d(u, Tu) = 0$, that is $u = Tu$. Similarly, we can prove that $Su = u$. Therefore, u is a common fixed point of S, T, I and J. The uniqueness of the common fixed point is obvious due to implicit condition (ii).

This completes the prove of Theorem 2.1.

As an application of this Theorem 2.1, we have the following common fixed point theorem for four families of mappings.

Theorem 2.2. Let $\{S_1, S_2, \ldots, S_m\}$, $\{T_1, T_2, \ldots, T_n\}$, $\{I_1, I_2, \ldots, I_p\}$, and $\{J_1, J_2, \ldots, J_q\}$ be four *families of self-mappings of a metric space* (X, d) *with* $S = S_1S_2 \ldots S_m$, $T = T_1T_2 \ldots T_n$, $I = I_1I_2 \ldots I_p$, *and* $J = J_1J_2 \ldots J_q$ *satisfying the following conditions*:

(iii) $S(X) \subset J(X)$ *and* $T(X) \subset I(X)$ *and*

(iv) $F(d(Sx, Ty), d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)) \le 0$,

for each x, y *in* X.

If one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is a complete space of X, then (S, I) or (T, J) have a point of coincidence.

Moreover, if $S_i S_j = S_j S_i$; $I_k I_l = I_l I_k$; $T_r T_s = T_s T_r$; $J_t J_u = J_u J_t$; $S_i I_k = I_k S_i$;
$$I_k T_r = T_r I_k\,;\ T_r J_t = J_t T_r\,;\ S_i J_t = J_t S_i\,;\ S_i T_r = T_r S_i \text{ and } J_t I_k = I_k J_t,$$

for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, p\}$, $r, s \in I_3 = \{1, 2, \ldots, n\}$, and

$t, u \in I_4 = \{1, 2, \ldots, q\}$, then , for all $i \in I_1$, $k \in I_2$, $r \in I_3$ and $t \in I_4$, the mappings S_i, I_k, T_r and J_t have a common fixed point.

Proof: Since the mappings S, T, J and I satisfy all the required relevant conditions of Theorem 2.1, so the pair (S, I) or (T, J) have a point of coincidence. Also, appealing to component wise commutativity of various pairs, we can prove that $SI = IS$ and $TJ = JT$ and hence obviously both the pairs (S, I) and (T, J) are coincidentally commuting. Moreover, all the conditions of Theorem 2.1 (for mappings S, T, I and J) are satisfied ensuring the existence of unique common fixed point z.

Now, we need to show that z remains the fixed point of all component mappings. For this, we consider

$$\begin{aligned} S(S_i z) &= ((S_1 S_2 \ldots S_m) S_i)z \\ &= (S_1 S_2 \ldots S_{m-1}) ((S_m S_i) z) = (S_1 S_2 \ldots S_{m-1}) (S_i S_m z) \\ &= (S_1 S_2 \ldots S_{m-2}) (S_{m-1} S_i (S_m z)) = (S_1 S_2 \ldots S_{m-2}) (S_i S_{m-1} (S_m z)) \\ &= (S_1 S_2 \ldots S_{m-3}) (S_{m-2} S_i (S_m z)) = \ldots = (S_i S_1 S_2 \ldots S_m) z \\ &= S_i (Sz) = S_i z. \end{aligned}$$

Similarly, we can prove

$$S(I_k z) = I_k(Sz) = I_k z;$$

$$I(I_k z) = I_k(Iz) = I_k z; \quad S(T_r z) = T_r(Sz) = T_r z; \quad S(J_t z) = J_t(Sz) = J_t z;$$

$$I(T_r z) = T_r(Iz) = T_r z; \quad I(J_t z) = J_t(Iz) = J_t z; \quad T(S_i z) = S_i(Tz) = S_i z;$$

$$T(T_r z) = T_r(Tz) = T_r z; \quad T(I_k z) = I_k(Tz) = I_k z; \quad T(J_t z) = J_t(Tz) = J_t z;$$

$$J(S_i z) = S_i(Jz) = S_i z; \quad J(T_r z) = T_r(Jz) = T_r z; \quad J(I_k z) = I_k(Jz) = I_k z;$$

$$J(J_t z) = J_t(Jz) = J_t z; \text{ and } I(S_i z) = S_i(Iz) = S_i z;$$

This shows that (for all i, r, k and t), $S_i z$, $T_r z$, $I_k z$ and $J_t z$ are other fixed points of S, I, T and J. Now, appealing to the uniqueness of common fixed points of S, T, I, and J, we have , for all i, r, k and t,

$$z = S_i z = T_r z = I_k z = J_t z,$$

which shows that z is a common fixed point of S_i, T_r, I_k and J_t for all i, r, k and t.

This completes the proof of theorem 2.2.

We now have the following corollaries related to above theorems.

Corollary 2.1. *By choosing S, T, I and J suitably and modifying the remaining hypotheses accordingly, the derived conclusions of Theorem* 2.1 *remain true if for all x, y in X and $F \in \Phi$, the implicit condition* (ii) *is replaced by any one of the following conditions*:

(A) $F(d(Sx, Sy), d(Ix, Jy), d(Jx, Sx), d(Jy, Sy), d(Ix, Sy), d(Jy, Sx)) \le 0$,
(derived by setting $S = T$)

(B) $F(d(Sx, Ty), d(Ix, Iy), d(Ix, Sx), d(Iy, Ty), d(Ix, Ty), d(Iy, Sx)) \le 0$,
(derived by setting $I = J$)

(C) $F(d(Sx, Sy), d(Ix, Iy), d(Ix, Sx), d(Iy, Sy), d(Ix, Sy), d(Iy, Sx)) \le 0$,
(derived by setting $S = T$ and $I = J$)

(D) $F(d(Sx, Sy), d(x, y), d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)) \leq 0$,

(derived by setting $S = T$ and $I = J$ = an identity map)

(E) $F(d(Sx, Ty), d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)) \leq 0$,

(derived by setting $I = J$ = an identity map)

Also, by setting $m = n = p = q$ and for all i, r, k and t, consider $S_i = T_r = I_k = J_t = F$ (say), then we have the following corollary as a variant of Bryant's Theorem [1].

Corollary 2.2.[1] *Let F be a self-mapping of a metric space (X, d) such that there exists some $n \in N$ satisfying*

$$F(d(F^n x, F^n y), d(x, y), d(x, F^n x), d(y, F^n y), d(x, F^n y), d(y, F^n x)) \leq 0,$$

for all x, y in X and $F \in \Phi$. If $F^n(X)$ is a complete subspace of X, then F has a unique fixed point.

[1] Examples

We now give the following examples to illustrate the above theorems.

Example 3.1. Consider $X = [0,6]$ with the usual metric. Define self-mappings S, T, I and J on X as

$$S\,0 = 0, \quad Sx = 1, \text{ for } 0 < x \leq 6,$$

$$T\,0 = 0, \quad Tx = 3, \text{ for } 0 < x \leq 6,$$

$$I\,0 = 0, \quad Ix = 5, \text{ for } 0 < x < 6, \quad I\,6 = 1,$$

and

$$J\,0 = 0, \quad Jx = 6, \text{ for } 0 < x < 6, \quad J\,6 = 1.$$

Then all four maps S, T, I and J are discontinuous, even at their unique common fixed point at $x = 0$. Also, the pairs (S, I) and (T, J) commute at $x = 0$ which is their common point of coincidence. Clearly, $S(X) = \{0, 1\} \subset \{0, 1, 6\} = J(X)$ and $T(X) = \{0, 3\} \subset I(X) = \{0, 3, 5\}$. If we define a continuous function $F : (R^+)^6 \to R$ by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, (t_5 + t_6)/2\},$$ where $k \in (0,1)$ and F satisfies

F_1, F_2. Also, F satisfies the implicit contractive condition (ii) for $k = \frac{1}{10}$.

Moreover, the pairs (S, I) and (T, J) are weakly commuting [15] and hence compatible [12] because,

$$|\,SI6 - IS6\,| = |\,1 - 5\,| > 0 = |\,I6 - S6\,| \text{ whereas}$$

$$|\,TJ6 - JT6\,| = |\,3 - 6\,| > |\,1 - 3\,| = |\,J6 - T6\,|.$$

Example 3.2. *Let $X = \{0, 1, ½, 1/2^2, 1/2^3, \ldots, \ldots\}$ be a metric space with the usual metric $d(x, y) = |\,x - y\,|$ for all x, y in X. For $n = 0, 1, 2, 3, \ldots$, define mappings $S, I : X \to X$ by*

$$S(0) = 1/2^2, \qquad S(1/2^n) = 1/2^{n+2},$$

$$I(0) = ½, \qquad I(1/2^n) = 1/2^{n+1} \text{ respectively.}$$

Also, we set $S = T$ and $I = J$. Then, clearly $S(X) = \{1/2^2, 1/2^3, \ldots\} \subset \{1/2, 1/2^2, 1/2^3, \ldots\} = I(X)$.

Define a continuous function $F : (R^+)^6 \to R$ by

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - a t_2^2 - b(t_3^2 + t_4^2 + 1),$$ with $a = ½$ and $b = ¼$, then F satisfies

F_1 and F_2. Furthermore,

$$F(d(S0, S1), d(I0, I1), d(I0, S0), d(I1, S1), d(I0, S1), d(I1, S0))$$
$$= F(0, 0, ¼, ¼, ¼, ¼) = -1/72 < 0.$$

Similarly, we can show that

$$F(d(S0, S1/2), d(I0, I1/2), d(I0, S0), d(I1/2, S1/2), d(I0, S1/2), d(I1/2, S0)) < 0,$$
$$F(d(S0,S1/4), d(I0,I1/4), d(I0,S0), d(I1/4,S1/4), d(I0,S1/4), d(I1/4,S0))\ 0,$$ and so on.

Also, for $x = 1/2^n$ and $y = 1/2^m$, for $n, m = 0,1,2, \ldots$ and $n \neq m$, we have

$$F(d(S1/2^n, S1/2^m), d(I1/2^n, I1/2^m), d(I1/2^n, S1/2^n), d(I1/2^m, S1/2^m),$$
$$d(I1/2^n, S1/2^m), d(I1/2^m, S1/2^n)) \leq 0.$$

Hence, all the conditions of Theorem 2.1 are satisfied except the completeness of the subspaces $S(X)$ and $T(X)$. Note that the mappings S and I have no point of coincidence. and even they are not continuous at the origin. This example shows that the completeness of the space is not sufficient for the existence of coincidence point, as the space X is complete with the usual metric.

Remarks: Theorem 2.2 is the application of Theorem 2.1. Also, the main result of Jeong and Rhoades [5] is a particular case of Theorem due to Imdad and Khan [3], which is established for six mappings. Since a variant of fixed point theorems corresponding to implicit conditions (D) and (E) appear in Popa, so our results extends the results of Popa[12, 13], improves the results of S. Kumar [8] & Popa [14], and also generalizes the result of Bryant [1] with respect to corollary 3.2.

REFERENCES

[1] V. W. Bryant, A remark on fixed point for iterated mappings, *Amer. Math. Monthly*, **75**(1968), 399–400.

[2] B. C. Dhage, On common fixed points of coincidentally commuting mappings in D-metric spaces, *Indian J. Pure Appl. Math.*, **30**(4)(1999), 395–406.

[3] M. Imdad and Q.H. Khan, Six mappings satisfying a rational inequality, *Rad. Mat.*, 9(1999), 251–260.

[4] M. Imdad, S. Kumar and M.S. Khan, Remarks on some fixed point theorem satisfying implicit relations, *Rad. Mat.*, **11**(2002), 135–143.

[5] G. S. Jeong and B.E. Rhoades, Some remarks for improving fixed point theorems for more that two mappings, *Indian J. Pure Appl. Math.*, **28**(9)(1987), 1177–1196.

[6] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.*, 9(4)(1986), 771–779.

[7] G. Jungck and B.E. Rhoades, Fixed points for set-valued functions without continuity, *Indian J. Pure Appl. Math.*, **29**(3)(1998), 227–238.

[8] S. Kumar and M. Imdad, A general common fixed point theorem without continuity, *J. Indian Acad. Math.*, **27**(2)(2005), 215–229.

[9] S. Kumar, Remarks on certain selected fixed point theorems II, *J. Kath. Univ. Sci. Engg. Tech.*, 1(4)(2007), 1–4.

[10] R. P. Pant, Common fixed points for four mappings, *Bull. Cal. Math.Soc.*, **90**(1998), 281–286.

[11] V. Popa, Fixed point theorems for implicit contractive mappings, *Studii Si Cercetari Stiintifice Matematica Seria (Bacau)*, 7(1997), 127–133.

[12] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstratio Math.*, **32**(1)(1999), 157–163.

[13] V. Popa, Some fixed point theorems for weakly compatible mappings, ***Rad. Mat.***, **10**(2001), 245–252.

[14] V. Popa, A generalization of Meir-Keeler type common fixed point theorem for four non-continuous mappings, ***Sarajevo J. Math.***, **1**(13)(2005), 135–142.

[15] S. Sessa, On a weak commutativity condition in fixed point considerations, ***Publ. Inst. Math. (Beograd)***, **32**(46)(1982), 146–153.