The Nepali Math. Sc. Report
Vol. 23, No. 2, 2004.

Semi Symmetric Non-Metric Connection on a Manifold with Generalised HSU-Structure

RAM NIVAS AND GEETA VERMA

Abstract : Semi symmetric metric connection have been studied by various mathematician including Yano [2], Mishra [3], Imai [4], S. I. Hussain [5]. M. D. Upadhaya and Jaya Pant [7], etc., manifold. Recently Nirmala S. Agashe and others [1] have defined the notion of semi symmetric non-metric connection on a Riemannian manifold. Singh and Nivas [6] studied semi-symmetric non-metric connection on almost para contact metric manifold. In this paper we study semi-symmetric non-metric connection on manifold with generalised Hsu-structure. In the first section, I have studied Nijenhuis tensor and integrability conditions of such manifolds. Some interesting results have been established in other sections of the paper

1. Preliminaries

Let an n-dimensional differentiable manifold M^n of class C^∞ admits a C^∞ tensor field F of the type $(1,1)$ a C^∞ a vector field T and C^∞ 1-form A such that

(1.1)

(i) $\overline{\overline{X}} = a^r X + A(X)T,$

(ii) $\overline{X} = F(X),$

(iii) $A(T) = -a^r,$

(iv) $A(FX) = 0,$

(v) $FT = 0,$

(vi) $g(T,X) = A(X),$

and

(vii) $g(\overline{X},\overline{Y}) = -a^r\, g(X,Y) - A(X)\,A(Y)$

where g is a non-singular metric tensor and 'a' is any non-zero complex number. Let us call such a structure a generalised almost contact metric structure

It follows from (1.1). (vii)

$$g(\overline{\overline{X}},\overline{\overline{Y}}) = -a^r\, g(\overline{X},\overline{Y})$$

Let us define

(1.2) $$F(X,Y) = g(FX,Y)$$

barring X in (1.2) we have

(1.3) $$F(\overline{X},Y) = g(F^2X,Y)$$

which by virtue of equation (1,1) (vii) yields

(1.4) $$F(\overline{X},Y) = a^r g(X,Y) + A(X)\,A(Y)$$

Now barring Y in (1.3) we have

$$F(X,\overline{Y}) = -\,a^r g(X,Y) - A(X)\,A(Y)$$

or

(1.5) $$F(X,\overline{Y}) = -\,(a^r g(X,Y) + A(X)\,A(Y))$$

Thus from (1.4) and (1.5) we have

$$F(X,\overline{Y}) + F(\overline{X},Y) = 0$$

Replacing X by T in equation (1.2) and making use of equation (1.1) (v) we obtain

(1.6) $$F(T,Y) = 0$$

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor

$$S(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

satisfied the formula

(1.7) $$S(X,Y) = A(Y)X - A(X)Y$$

∇ is said to be semi-symmetric non-metric with respect to the associated metric g if

(1.8) $$\nabla_X\, g(Y,Z) = -\,A(Y)\,g(X,Z) - A(Z)\,g(X,Y)$$

We define ∇ to be semi-symmetric non-metric F-connection if in addition (1.7), (1.8) ∇ satisfies

(1.9) $$(\nabla_X\, F) = 0$$

Suppose ∇ is a Riemannian connection on M^n then we can always put [1]

(1.10) $$\overset{*}{\nabla}_X Y = \nabla_X Y + U(X,Y)$$

U being tensor of type (1.2) satisfying

(1.11) $$g(U(X,Y)Z) + g(U(X,Z)Y) = A(Y)\,g(X,Z) + A(Z)\,g(X,Y)$$

obviously we have

(1.12) $$S(X,Y) = U(X,Y) - U(Y,X)$$

Nirmala S. Agashe and others expressed the value of $U(X,Y)$ in terms of S and S'

(1.13) $$U(X,Y) = \tfrac{1}{2}\,[S(X,Y) + S'(X,Y) + S'(Y,X)] + g(X,Y)T$$

where

(1.14) $$g(S(Z,X),Y) \underset{=}{\text{def}} g(S'(X,Y),Z)$$

It can be verified that

$$S'(X,Y) = A(X)Y - g(X,Y)T$$

and

(1.15) $$U(X,Y) = A(Y)X.$$

Thus we get

(1.16) $$\nabla_X Y = D_X Y + A(Y)X$$

It is easy to verify that

(1.17)

(i) $S'(Y,X) = U(X,Y) - g(X,Y)T,$

(ii) $g(S(X,Y),T) = 0,$

(iii) $S(X,T) = -\overline{\overline{X}},$

(iv) $S'(T,X) = U(X,T) + A(X)T,$

and

(v) $S'(X,Y) - S'(Y,X) = S(X,Y)$

Theorem 1.1. *In a generalised Hsu-structure manifold M^n the torsion tensor of the semi-symmetric non-metric connection satisfies the following identities.*

(1.18)

(i) $S(X,T) = -\overline{\overline{X}},$

(ii) $S(\overline{X},T) = -a^r\,\overline{X},$

(iii) $S(\overline{\overline{X}},Y) = a^r A(Y)X + A(X)A(Y)T,$

(iv) $S(\overline{\overline{X}},Y) + S(X,\overline{\overline{Y}}) = a^r S(X,Y).$

(v) $S(\overline{\overline{X}},T) = a^r S(\overline{X},T) = a^{2r}\,\overline{X},$

(vi) $A(S(X,Y)) = 0,$

(vii) $\overline{\overline{S(X,Y)}} = a^r S(X,Y).$

Now we will establish certain identities among the (0,3) type tensor defined as follows

(1.19)

(i) $S'(X,Y,Z) \underset{=}{\text{def}} g(S(X,Y),Z)$

(ii) $U'(X,Y,Z) \underset{=}{\text{def}} g(U(X,Y),Z)$

or equivalently

$$S'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(X,T) \\ g(Y,Z) & g(X,Z) \end{vmatrix}$$

and

$$U'(X,Y,Z) = \begin{vmatrix} g(Y,T) & g(Z,T) \\ g(X,Y) & g(X,Z) \end{vmatrix}$$

Theorem 1.2. *The following relations hold in a generalised Hsu-structure non-metric manifolds*

(1.20)

(i) $S(X,Y,\overline{\overline{Z}}) = a^r S'(X,Y,Z)$

(ii) $U'(\overline{\overline{X}},Y,Z) = -a^r U'(X,Y,Z) = 0$

(iii) $U'(\overline{\overline{X}},\overline{Y},\overline{Z}) = S'(\overline{X},\overline{Y},\overline{\overline{Z}}) = 0$

(iv) $S'(Z,Y,\overline{\overline{X}}) = a^r U'(X,Y,Z) = 0$

(v) $U'(\overline{\overline{Z}},Y,X) = a^r S'(X,Y,Z)$

(vi) $S'(X,Y,\overline{\overline{Z}}) = U'(\overline{\overline{Z}},Y,X)$

2. Nijenhuis Tensor

The Nijenhuir tensor is given by

(2.1) $$N(X,Y) = [\overline{X},\overline{Y}] + \overline{\overline{[X,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}$$

Making use of (1.1) (i) in (2.1) we get

(2.2) $$N(X,Y) = [\overline{X},\overline{Y}] + a^r [X,Y] + A([X,Y])T - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}]}$$

We now put

(2.3) $$B(X,Y) = [\overline{X},\overline{Y}] - \overline{[X,\overline{Y}]},$$

(2.4) $$H(X,Y) = [\overline{X},\overline{Y}] - \overline{[\overline{X},Y]},$$

(2.5) $$W(X,Y) = [\overline{X},\overline{Y}] - a^r [X,Y].$$

Theorem 2.1. *The Nijenhuis tensor and $B(X,Y)$ are related as*

(2.6) $$a^r B(X,Y) - B(X,\overline{\overline{Y}}) = a^r N(X,Y) - A(Y)\,\overline{[X,T]} + a^r A(Y)\,[X,T] + A(Y)A([X,T])T$$

Proof: Barring Y in (2.3) and making use of (1.1) (i) we get

$$(2.7) \qquad B(X,\overline{Y}) = a^r(\overline{X},Y) + A(Y)\,[\overline{X},T] - a^r\,\overline{[\overline{X},T]} + \overline{A(Y)[X,T]}$$

Again barring above equation and making use of (1.1) (i) we get

$$(2.8) \qquad \overline{B[X,\overline{Y}]} = a^r\,\overline{[\overline{X},Y]} + A(Y)\overline{[\overline{X},T]} - a^{2r}[X,Y]$$
$$- a^r A([X,Y])T - a^r A(Y)\,[X,T]$$
$$- A(Y)\,A([X,T])T$$

Now from equation (2.3) and (2.8) we obtain

$$(2.9) \qquad a^r\,B(X,Y) - \overline{B(X,\overline{Y})} = a^r[\overline{X},\overline{Y}] - a^r\,\overline{(X,\overline{Y})}$$
$$- a^r\,\overline{[\overline{X},Y]} - a^{2r}[X,Y] - A(Y)\overline{[\overline{X},T]}$$
$$+ a^r A([X,Y])T + a^r A(Y)[X,T]$$
$$+ A(Y)\,A([X,T])T$$

Making use of (2.2) in (2.9) we get the result putting T for Y in (2.6) and making use of (1.1) (iii) and (1.1) (v) we have in a differentiable manifold.

$$(2.10) \qquad a^r\,B(X,T) = a^r N[X,T] + a^r[\overline{X},T]$$
$$- a^{2r}[X,T] - a^r A([X,T])T$$

Theorem 2.2. *In a differentiable manifold M^n we have*

$$(2.11) \qquad a^r H(X,Y) - H\,\overline{[\overline{X},Y]} = a^r N(X,Y) - A(X)\,\overline{[T,\overline{Y}]}$$
$$+ a^r A(X)\,[T,Y]$$
$$+ A(X)\,A([T,Y])T$$

Proof : Barring X in (2.4) and making use of (1.1) (i) we get

$$(2.12) \qquad H(\overline{X},Y) = a^r[X,\overline{Y}] + A(X)\,[T,\overline{Y}]$$
$$- a^r\,\overline{[X,Y]} - A(X)\,\overline{[T,Y]}$$

Now barring the whole equation (2.12) and making use of (1.1). (i) we have

$$(2.13) \qquad \overline{H(\overline{X},Y)} = a^r\,\overline{[X,\overline{Y}]} - A(X)\,\overline{[T,\overline{Y}]} - a^{2r}[X,Y]$$
$$- a^r A([X,Y])\,T - a^r A(X)\,[T,Y]$$
$$- A(X)\,A([T,Y])\,T$$

Now from (2.4) and (2.13) we have

$$(2.14)\qquad a^r H(X,Y) - \overline{H(\overline{X},Y)} = a^r[\overline{X},\overline{Y}] - a^r\overline{[X,\overline{Y}]}$$
$$- a^r[\overline{\overline{X},Y}] + a^{2r}[X,Y]$$
$$+ a^r A([X,Y])T - A(X)\,\overline{[T,\overline{Y}]}$$
$$+ a^r A(X)[T,Y] + A(X)A[T,Y]T$$

Thus, from (2.2) and (2.14) we obtained the required result.
Replacing X by T in (2.11) and using (1.1) (iii) and (1.1) (v) we get in generalised Hsu-structure almost contact metric manifold.

$$(2.15)\qquad a^r H(T,Y) = a^r N(T,Y) + a^r\,\overline{[T,\overline{Y}]}$$
$$- a^{2r}[T,Y] - a^r A([T,Y])T$$

Theorem 2.3. *In a generalised Hsu-structure almost contact metric structure manifold M^n we have*

$$(2.16)\qquad a^r W(X,Y) - \overline{W(\overline{X},Y)} = a^r N(X,Y) - a^r A([X,Y])T$$
$$- A(X)\,\overline{[T,\overline{Y}]}$$

Proof : Barring X in (2.5) and making use of (1.1) (i) we get

$$(2.17)\qquad W(\overline{X},Y) = a^r(X,\overline{Y}) + A(X)(T,\overline{Y})$$
$$+ a^r[\overline{X},Y]$$

Again barring (2.1) and making use of equation (1.1) (i) we have

$$(2.18)\qquad \overline{W(\overline{X},Y)} = a^r\,\overline{[X,\overline{Y}]} + A(X)\,\overline{[T,\overline{Y}]}$$
$$+ a^r[\overline{\overline{X},\overline{Y}}]$$

Thus with the help of (2.2), (2.5) and (2.18) we get (2.16). Replacing X by T in (2.16) and using the equation (1.1) (iii) and (1.1) (v) we can show that equation (2.16) is equivalent to

$$(2.19)\qquad a^r W(T,Y) = a^r N(T,Y) - a^r A([T,Y])T + a^r\,\overline{[T,\overline{Y}]}$$

3. The Curvature Tensor

The curvature tensors of semi-symmetric non-metric connection ∇ and the Riemanian connection D are respectively represented by R & K as follows

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$K(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

Thus we state the following theorem

Theorem 3.1. *The two curvature tensor are related to the following equations*

$$R(X,Y)Z = K(X,Y)Z + A(D_Y\ Z)X - A(D_X Z)Y - A(Z)\,S(X,Y)$$
$$+ X(A(Z))Y - Y(A(Z))X$$

4. Integrability Condition

Theorem 4.1. *In order that a generalised almost contact metric manifold to be completely integrable it is necessary that*

$$A([\bar{X},\bar{Y}])T = 0$$

Proof: Barring X in (2.2) and with the help of equation (1.1) (i) we get

From equation (2.2) and (4.2) we have

$$(4.3) \qquad \overline{N(\bar{X},Y)} + a^r N(X,Y) = A(X)\,\overline{[T,\bar{Y}]} - A\,[\bar{X},\bar{Y}]T$$
$$- a^r A(X)\,[T,Y] - A(X)\,([T,Y])T$$

Using

$$(4.4) \qquad N[T,Y] = a^r (T,Y) + A([T,Y])T - \overline{[T,\bar{Y}]}$$

in (4.3) we have

$$(4.5) \qquad N(\bar{\bar{X}},Y) + a^r N(X,Y) = -A(X)\,N(T,Y)$$
$$- A([\bar{X},\bar{Y}])T$$

For completely integrable manifold equation (4.5) reduces to give result in theorem.

Theorem 4.2. *For a completely integrable generalised almost contact metric structure manifold we have*

$$(4.6) \qquad A(X)\,\{[T,\bar{Y}] - \overline{[\bar{T},Y]}\} + A([\bar{X},Y])T$$
$$= A(X)\,\{[\bar{X},T] - \overline{[X,\bar{T}]}\} + A([X,\bar{Y}])T$$

Proof: Barring X and Y in equation (2.2) and making use of (1.1) (i) we get the following equations

$$(4.7) \qquad N(\bar{X},Y) = a^r [X,\bar{Y}] + A(X)\,[T,\bar{Y}]$$
$$+ a^r [\bar{X},Y] + A([\bar{X},Y])\,T - \overline{[\bar{X},\bar{Y}]}$$
$$- a^r\,\overline{[X,Y]} - A(X)\,\overline{[T,Y]}$$

$$(4.8) \qquad N(X,\overline{Y}) = a^r[\overline{X},Y] + A(Y)[\overline{X},T] + a^r[X,\overline{Y}] + A([X,\overline{Y}])T - a^r\overline{[\overline{X},Y]} - [\overline{\overline{X},\overline{Y}}] - A(Y)\overline{[X,T]}$$

Now from these two equations (4.7) and ((4.8) and using $N(X,Y)$ we have then required result (4.6).

REFERENCES

[1] Agashe, N. S., and Chaffe, M. (1992) : *A semi symmetric non-metric connection on a Riemannian manifolds*, Indian Journal of Pure and Applieu Math Vol. 23(6), pp. 399–409.

[2] Hussain, S. I., (1976) : *Semi symmetric metric connection on almost contact manifolds tensor*, N. S., Vol. 80, pp. 133–139.

[3] Iami, T., (1972) : *Notes on semi-symmetric metric connections*. Tensor N. S. Vol. 24, pp. 293–296.

[4] Mishra, R.S., (1972) : *Affine connection in an almost Grayan manifold* Tensor, N. S., Vol. 23. Pp. 317–322.

[5] Singh, J. P. (1997) : *On almost para contact metric manifolds with semi symmetric non-metric connection*. Ph.D. Thesis, Lucknow University,

[6] Yano, K. (1970) : *On semi symmetric metric connection*. Rev. Roumaine Math. Pure & Appl. Vol. 15, pp 1579–1586.

1. **RAM NIVAS**
Department of Mathematics and Astronomy
University of Lucknow,
Lucknow, (U.P) India.

2. **GEETA VERMA**
Department of Applied Sciences
Azad Institute of Engineering and Technology,
Natkur, Post-Chandrawal,
Bangla Bazar Road, Locknow – 226002.
(U.P.) India.