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Some Operation–Transform Formulas For S_μ–Transform

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1. Introduction:

In an earlier Paper [1] S_μ-Transform has been defined by

$$(1.1) \qquad S_\mu[f(t)] = F(s) = \int_0^\infty f(t)\frac{e^{-\mu}(s/t)}{s+t}dt \qquad (\mu \geq 0).$$

where $f(t)$ is a suitably restricted conventional function defined on the +ve real line $0 < t < \infty$ & $0 < \text{Res} < \infty$. It has been generalised in the case of generalised functions as

$$(1.2) \qquad S_\mu[f(t)] = F(s) = \langle f(t), \frac{e^{-\mu}(s/t)}{s+t} \rangle \qquad (\mu \geq 0).$$

Its invention formula has been also derived. Here it is proposed to discuss some operation transform formulae of the transform given by (1.1)

2. Operation –Transform Formulae for S_μ-Transform

Differentiation : If $\phi \in S_\mu$, where B_μ is the space of all complex valued smooth functions $\phi(t)$ such that for each $\phi(t) \in B_\mu$, we have

$$\rho_n(\phi) = \sup_{0<t<\infty} |D^n\phi(t)| \qquad (n = 0, 1, 2 \ldots\ldots)$$

bounded.

We shall prove that

$$(2.1) \qquad \rho_n[-D\phi] = \rho_{n+1}[\phi]$$

Since,

$$\rho_n[-D\phi] = \sup_{0<t<\infty} |D^n(-D\phi)|$$

$$= \sup_{0<t<\infty} |D^{n+1}(-D\phi)|$$

$$= \rho_{n+1}[\phi].$$

Therefore, we get

$$\rho_n[-D\phi] = \rho_{n+1}[\phi].$$

From (2.1) it follows that $\phi \to -D\phi$ is a continuous and linear mapping of β_μ onto itself. Therefore, from Theorem 1.10–1 due to Zemanian [2, p.29], the adjoint mapping $f \to Df$ is also a continuous and linear mapping of β'_μ on to itself where β'_μ is the dual of β_μ and we get

(2.2) $$\langle Df(t), \phi(t)\rangle = \langle f(t), -D\phi(t)\rangle.$$

Now, we prove that the following operation–transform formula

(2.3) $$S_\mu[D^n f] \le K.S_\mu[|f(t)|].$$

Proof: Using, the generalised definition of S_μ–transform and the relation (2.2), we get

$$S_\mu[D^n f] = \langle D^n f(t)\,\frac{e^{-\mu s/t}}{s+t}\rangle$$

$$= \langle f(t), (-D)^n f\,\frac{e^{-\mu s/t}}{s+t}\rangle$$

$$= \langle f(t), \sum_{v=0}^{n} n_{c_v}(-D)^{n-v}\, e^{-\mu}\, s/t(-D)^v \frac{1}{s+t}\rangle$$

Therefore, we get

(2.4) $$S_\mu[D^n f(t)] = \langle f(t), \frac{e^{-\mu s/t}}{s+t}\cdot\frac{P_n(t)}{Q_n(t)}\rangle,$$

where $P_n(t)$ and $Q_n(t)$ are the polynomials in t such that order of $Q_n(f) \ge$ order of $P_n(t)$. Let us suppose that f is a regular generalised function of β'_μ. Therefore, for $\phi \in \beta_\mu$, we have

$$\langle f, \phi\rangle = \int_0^\infty f(t)\,\phi(t)\,dt$$

and

$$|\langle f, \phi\rangle| \le \int_0^\infty |f(t)|\;|\phi(t)|\,dt$$

Consequently, we get

(2.5) $$|\langle f, \phi\rangle| \le \langle |f|, |\phi|\rangle$$

An appeal to (2.4) and (2.5) given

$$S_\mu[D^n f] \le \langle |f(t)|, \left|\frac{e^{-\mu s/t}}{s+t}\right|\left|\frac{P_t(t)}{Q_n(t)}\right|\rangle$$

$$\le \langle |f(t)|, \left|\frac{e^{-\mu s/t}}{s+t}\right|\rangle, K\rangle,$$

where $\left|\dfrac{P_n(t)}{Q_n(t)}\right| \le K$ (cont.) ; $0 < t < \infty$; $0 < s < \infty$ & $\mu \ge 0$.

Therefore, we get

$$S_\mu |D^n f| \le K \left\langle |f(t)|, \left|\frac{e^{-\mu s/t}}{s+t}\right| \right\rangle$$

$$\le K \,.\, S_\mu [\,f(t)\,|\,].$$

This completes the proof.

Multiplication by an Exponential Functions

Let μ be a real number such that $\mu \ge 0$. Now we prove that $\phi(t) \to e^{-\mu t}\,\phi(t)$ is a continuous and linear mapping from B_μ on to itself.

Proof : Let $\phi \in B_\mu$, We have

$$D^n[e^{-\mu t}\phi(t)] = \sum_{\nu=0}^{n} n_{c_\nu} D^{n\nu}\, e^{-\mu t}\, D^\nu\, \phi(t)$$

$$= \sum_{\nu=0}^{n} n_{c_\nu}\, (-\mu)^{n-\nu}\, e^{-\mu t}\, D^\nu\, \phi(t)$$

Therefore, we get

$$|D^n[e^{-\mu t}\phi(t)]| \le \sum_{\nu=0}^{n} K\,|D^\nu\, \phi(t)|,$$

where $|n_{c_\nu}(-\mu)^{n-\nu}\, e^{-\mu t}| \le K$ for $\mu \ge 0$ & $0 < t < \infty$.

Thus we get

(2.6) $\qquad \rho_n[e^{-\mu t}\,\phi(t)] \le K \sum_{\nu=0}^{n} |\rho_n|\,\phi(t)| \qquad (n = 0, 1, 2, \ldots\,;\ \nu = 0, 1, 2, \ldots).$

From (2.6), it follow that $\phi(t) \to e^{-\mu t}$ is a continuous and linear mapping of B_μ on to itself. Therefore, from Theo. 1.10–1 due to Zemanian [2, p.29] the adjoint mapping $f \to e^{-\mu t} f$ is also a continuous and linear mapping of B_μ on to itself and we get.

(2.7) $\qquad \langle e^{-\mu t} f(t), \phi(t)\rangle = \langle f(t), e^{-\mu t}\,\phi(t)\rangle.$

An appeal to (2.7) & the generalised definition of S_μ–transform.

We get

$$S_\mu[e^{-\mu t} f(t)] = \left\langle e^{-\mu t} f(t), \frac{e^{-\mu s/t}}{s+t} \right\rangle$$

$$= \left\langle f(t), e^{-\mu t}\frac{e^{-\mu s/t}}{s+t} \right\rangle$$

Therefore,

$$|S_\mu[e^{-\mu t} f(t)]| \le \left\langle |f(t)|, |e^{-\mu t}| \left|\frac{e^{-\mu s/t}}{s+t}\right| \right\rangle$$

by (2.5) if f is a regular generalised function

$$\leq M \langle |f(t)|, \frac{e^{-\mu s/t}}{s+t} \rangle$$

$$\leq M S_\mu[|f(t)|],$$

where $|e^{-\mu t}| \leq M$

Thus we get an operation-transform formula

$$S_\mu |[e^{-\mu t} f(t)]| \leq M S_\mu[|f(t)|] \quad (2.8)$$

Multiplication by $(S+t)^{-\lambda}$ where $\lambda \to 0; 0 < t < \infty$ & $0 < s < \infty$.

We prove that $\phi(t) \to (s+t)^{-\lambda} \phi(t)$ is an a continuous and linear mapping of B_μ on itself, where $\lambda > 0$; $0 < t < \infty$ & $0 < s < \infty$.

Proof: Let $\phi \in B_\mu$, We have

$$D_n[(s+t)^{-\lambda} \phi(t)] = \sum_{v=0}^{n} n_{c_v} D^{n-v} (s+t)^{-\lambda} D^v \phi(t)$$

$$= \sum_{v=0}^{n} n_{c_v} (-\lambda)(-\lambda-1)\ldots(-\lambda-(n-v-1)(s+t)^{-\lambda-v+v} D^v \phi(t)$$

Therefore, we get

$$|D^n (s+1)^{-\lambda} \phi(t)| \leq M \sum_{v=0}^{n} |D^v \phi(t)|,$$

where $\quad |n_{c_v} (--\lambda) (-\lambda-1) \ldots (-\lambda-n+v+1) (s+t)^{-\lambda-n+v}| \leq M.$

$(n = 0, 1, 2, \ldots\ldots;\ v = 0, 1, 2, \ldots\ldots)$

Thus we get

$$\rho_n[(s+t)^{\mu} \phi(t)] \leq M \sum_{v=0}^{n} \rho_v [\phi(t)] \quad (2.9)$$

From (2.9) it follows that $\phi(t) \to (S+t)^{-\lambda} \phi(t)$ is a continuous & linear mapping of B_μ on to itself. Therefore, from Theo. 1.10–1 due to Zemainan [2.p.29], the adjoint mapping $f \to (s+t)^{-\lambda} f$ of $\phi \to (s+t)^{-\lambda} \phi$ is also a continuous and linear mapping of β'_μ on itself and we get

$$\langle (s+t)^{-\lambda} f(t)\, \phi(t) \quad f(t)\,(s+t)^{-\lambda} \phi(t) \rangle \quad (2.10)$$

An appeal to (2.10) & the generalised definition of S_μ–transform gives

$$S_\mu [(s+t)^{-\mu} f(t)] = \langle (s+t)^{-\lambda} f(t),\ e^{-\mu s/t} \rangle$$

$$= \langle f(t), (s+t)^{-\lambda} \frac{e^{-\mu s/t}}{s+t} \rangle$$

If f be a regular generalised function then by using (2.5), we get

$$|S_\mu[(s+t)^{-\lambda} f(t)]| \leq \langle |f(t)|, |(s+t)^{-\lambda}| \left| \frac{e^{-\mu s/t}}{s+t} \right| \rangle$$

$$\leq N \langle |f(t)|, \frac{e^{-\mu s/t}}{(s+t)} \rangle \leq N\, S_\mu[|f(t)|],$$

where $|(s+t)^{-\lambda}| \leq N$

Thus we get an operation-transform formula

(2.11) $$|S_\mu[(s+t)^{-\lambda} f(t)]| \leq NS_\mu \langle |f(t)|]$$

Sifting: Let T be a fixed real number such that $0 < t + T < \infty$ & $0 < t < \infty$. Let $\phi(t) \in \beta_\mu$. Now we will prove that $(t + T)$ is a continuous & linear mapping of β_μ on to itself.

Proof: Let us consider

$$D^n[\phi(t+T)] = (d/dt)^n\, |\phi(t+T)|$$

$$= \left[\frac{d}{d(t+T)} \frac{d(t+T)}{dt}\right]^n |\phi(t+T)|$$

$$= \left(\frac{d}{d(t+T)}\right)^n |\phi(t+T)|$$

$$= D^n_{t+T}\, \phi(t+T)$$

$$= D^n_{t_1}[\phi\, t_1], [t_1 = t + T],$$

where $0 < t + T < \infty$ and $0 < t < \infty$.

Therefore, we get

(2.12) $$D^n_t[\phi(t+T) = D^n_t[\phi(t)], |t_1 = t|$$

i.e. $$\rho_n |[\phi(t+T)]| = \rho_1 |\phi(t)|$$

Thus from (2.12), it follows that $\phi(t) \to \phi(t+T)$ is a continuous and linear mapping of β_μ on to itself. Its inners mapping $\phi(t) \to \phi(t+T)$ is also a continuous and linear mapping of β_μ on to itself. Therefore, $\phi(t) \to \phi(t+T)$ is an isomorphism of β_μ onto itself. The adjoint mapping of $\phi(t) \to \phi(t+T)$ is f(t) $\to f(t+T)$ which is also a continuations and linear mapping of β'_μ onto itself due to Theorem 1.10–1 of Zemanian [2.p.29] and we get

(2.13) $$\langle f(t+T), \phi(t) \rangle = \langle f(t), \phi(t+T) \rangle$$

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